

ÉTALE DEVISSAGE, DESCENT AND PUSH-OUTS OF STACKS

DAVID RYDH

ABSTRACT. We show that the push-out of an étale morphism and an open immersion exists in the category of algebraic stacks and show that such push-outs behave similarly to the gluing of two open substacks. For example, quasi-coherent sheaves on the push-out can be described by a simple gluing procedure. We then outline a powerful devissage method for representable étale morphisms using such push-outs. We also give a variant of the devissage method for representable quasi-finite flat morphisms.

INTRODUCTION

Let X be a scheme and let $X = U \cup V$ be an open cover of X . It is well-known that:

- (i) Many objects over X (such as quasi-coherent sheaves) correspond to objects over U and V with a gluing datum over $U \cap V$. No cocycle condition is needed as there are no non-trivial triple intersections.
- (ii) The scheme X is the push-out of the open immersions $U \cap V \rightarrow U$ and $U \cap V \rightarrow V$.
- (iii) Given two open immersions $W \subseteq U$ and $W \subseteq V$ of schemes we can glue these to a scheme $X = U \cup_W V$. The scheme X is the push-out of $W \subseteq U$ and $W \subseteq V$ and we recover W as the intersection of U and V .

In (i)–(iii), we can also replace “scheme” by “algebraic space” or “algebraic stack”. The purpose of this paper is to show that in the category of algebraic spaces or algebraic stacks we can further extend these results, taking one open immersion and one *étale* morphism instead of two open immersions. We also outline a powerful devissage method for étale morphisms based upon these results as well as an extension to quasi-finite flat morphisms.

The simplest open coverings are of the type $X = U \cup V$ discussed above and every open covering is a composition of such basic coverings. The devissage results explain how every étale (resp. quasi-finite flat) morphism is built up from étale neighborhoods and finite étale (resp. finite flat) coverings.

To be able to state our results we need to make “objects” in (i) above more precise. Usually this is done in the language of fibered categories and

Date: 2010-05-12.

2000 Mathematics Subject Classification. Primary 14A20; Secondary 14F20, 18A30, 18F20.

Key words and phrases. étale neighborhood, devissage, descent, push-out, algebraic stack, 2-sheaves.

Supported by the Swedish Research Council.

stacks. However as we need the base category to be the 2-category **Stack** it is more convenient to replace fibered categories and stacks with 2-*functors* and 2-*sheaves* as introduced by R. Street [Str82b, Str82a]. The literature on 2-sheaves is surprisingly meager and scattered. To not burden this, essentially geometrical, paper with a long categorical treatment of 2-sheaves, we have chosen to give a short comprehensible introduction in Appendix D which exactly covers what we need. Two examples of 2-sheaves to keep in mind are the 2-sheaf of quasi-coherent sheaves $\mathbf{QCoh}(-): \mathbf{Stack}^{\mathrm{op}} \rightarrow \mathbf{Cat}$ and the 2-sheaf $\mathbf{Hom}(-, Y): \mathbf{Stack}^{\mathrm{op}} \rightarrow \mathbf{Cat}$ for a given stack Y , cf. Appendix E.

Let X be an algebraic stack and let $Z \subseteq |X|$ be a closed subset. An *étale neighborhood* of Z is an étale morphism $f: X' \rightarrow X$ such that the restriction $f|_{Z_{\mathrm{red}}}: f^{-1}(Z_{\mathrm{red}}) \rightarrow Z_{\mathrm{red}}$ is an isomorphism. If $X = U \cup V$ is a union of two open substacks, then $V \rightarrow X$ is an étale neighborhood of $X \setminus U$. Note that we do not require that f is separated, nor representable, but étale signifies that f is at least represented by Deligne–Mumford stacks. We can now state the main theorems of this paper, generalizing (i)–(iii) in the beginning of the introduction.

Theorem A (Descent). *Let X be an algebraic stack and let $U \subseteq X$ be an open substack. Let $f: X' \rightarrow X$ be an étale neighborhood of $X \setminus U$ and let $U' = f^{-1}(U)$. Let $\mathbf{F}: \mathbf{Stack}_{\mathrm{\acute{e}t}/X}^{\mathrm{op}} \rightarrow \mathbf{Cat}$ be a 2-sheaf in the étale topology, cf. Appendix D. Then the natural functor*

$$(|_U, f^*): \mathbf{F}(X) \rightarrow \mathbf{F}(U) \times_{\mathbf{F}(U')} \mathbf{F}(X')$$

is an equivalence of categories.

Theorem B (Étale neighborhoods are push-outs). *Let*

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ f|_U \downarrow & \square & \downarrow f \\ U & \xrightarrow{j} & X \end{array}$$

be a cartesian diagram of algebraic stacks such that $j: U \rightarrow X$ is an open immersion and such that $f: X' \rightarrow X$ is an étale neighborhood of $X \setminus U$. Then X is the push-out of $f|_U$ and j' in the category of algebraic stacks, that is, the cartesian square is also co-cartesian.

Theorem C (Existence of push-outs). *Let X' be an algebraic stack, let $j': U' \rightarrow X'$ be an open immersion and let $f_U: U' \rightarrow U$ be an étale morphism. Then the push-out X of j' and f_U exists in the category of algebraic stacks. The resulting co-cartesian diagram*

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ f_U \downarrow & & \downarrow f \\ U & \xrightarrow{j} & X \end{array}$$

is also cartesian, j is an open immersion and f is an étale neighborhood of $X \setminus U$. Furthermore,

- (i) *The formation of the push-out commutes with arbitrary base change.*
- (ii) *If f_U is representable, then so is f .*
- (iii) *If j' is quasi-compact then so is j . If in addition X' and U are quasi-separated then so is X .*
- (iv) *If f_U is representable and X' and U have separated diagonals, then the diagonal of X is separated.*
- (v) *If X' and U are algebraic spaces, then so is X .*

(also see Proposition (2.4) for further properties)

For the applications in mind, e.g., the devissage method, it is useful to have Theorem C for étale morphisms f_U which are not representable and in this case X need not have separated diagonal, cf. Examples (2.5). It is thus natural to treat algebraic stacks with non-separated diagonals. On the other hand, the queasy reader is encouraged to assume that all algebraic stacks are at least quasi-separated, i.e., have quasi-compact and quasi-separated diagonals.

We will now state the étale devissage theorem. Let S be an algebraic stack. We let $\mathbf{Stack}_{\text{fp},\text{ét}/S}$ denote the 2-category of étale and finitely presented morphisms $X \rightarrow S$ and let $\mathbf{Stack}_{\text{repr},\text{sep},\text{fp},\text{ét}/S}$ denote the subcategory of morphisms that are representable and separated. The second category is equivalent to a 1-category.

Theorem D (Devissage). *Let S be a quasi-compact and quasi-separated algebraic stack and let \mathbf{E} be either $\mathbf{Stack}_{\text{fp},\text{ét}/S}$ or $\mathbf{Stack}_{\text{repr},\text{sep},\text{fp},\text{ét}/S}$. Let $\mathbf{D} \subseteq \mathbf{E}$ be a full subcategory such that*

- (D1) *if $X \in \mathbf{D}$ and $(X' \rightarrow X) \in \mathbf{E}$ then $X' \in \mathbf{D}$,*
- (D2) *if $X' \in \mathbf{D}$ and $X' \rightarrow X$ is finite, surjective and étale, then $X \in \mathbf{D}$, and*
- (D3) *if $j: U \rightarrow X$ and $f: X' \rightarrow X$ are morphisms in \mathbf{E} such that j is an open immersion and f is an étale neighborhood of $X \setminus U$, then $X \in \mathbf{D}$ if $U, X' \in \mathbf{D}$.*

Then if $(X' \rightarrow X) \in \mathbf{E}$ is representable and surjective and $X' \in \mathbf{D}$, we have that $X \in \mathbf{D}$. In particular, if there exists a representable and surjective morphism $X \rightarrow S$ in \mathbf{E} with $X \in \mathbf{D}$ then $\mathbf{D} = \mathbf{E}$.

Theorem D is generalized to quasi-finite flat morphisms in Section 6. Let us explain how Theorem D usually is applied. Suppose that we want to prove a statement $P(S)$ for an algebraic stack S and that we know that the corresponding statement $P(S')$ is true for some S' where $S' \rightarrow S$ is representable, étale and surjective. A typical situation is when S is a Deligne–Mumford stack and $S' \rightarrow S$ is a presentation. We let \mathbf{D} be the subcategory of $\mathbf{E} = \mathbf{Stack}_{\text{fp},\text{ét}/S}$ of stacks $X \rightarrow S$ such that $P(X)$ holds. It is then enough to verify conditions (D1)–(D3) for \mathbf{D} to deduce that $P(S)$ holds. If $S' \rightarrow S$ is also separated, then we can work in the smaller category $\mathbf{Stack}_{\text{repr},\text{sep},\text{fp},\text{ét}/S}$ but if we do not assume that $S' \rightarrow S$ is separated we have to include non-representable morphisms even though $S' \rightarrow S$ is representable.

For *algebraic spaces*, Theorems A–C are almost folklore. Parts of them or other closely related results appear in [RG71, §5.7], [BLR90, §6.2], [BL95] and [CLO09, §3.1]. The first aim of this paper is to state and prove Theorems A–C for *algebraic stacks*, a highly non-trivial task compared to the

case with algebraic spaces. The second aim is Theorem D which explains and generalizes the devissage method which is implicit in some of the above mentioned works.

The devissage method can be used to prove certain existence results that can be shown étale-locally. This includes Raynaud–Gruson’s flatification by blow-ups [RG71], tame étalification by stacky blow-ups and compactifications of tame Deligne–Mumford stacks [Ryd09b] and the existence of absolute noetherian approximation of stacks [Ryd09c]. We also expect the devissage method to be useful in applications of a completely different flavor.

Outline. In Section 1 we prove Theorems A and B. In Section 2 we describe some general properties of étale neighborhoods and in Section 3 we give a proof of Theorem C. In Section 4 we show that every constructible sheaf is locally constant on the stratification induced by an open filtration. Equivalently, a representable étale morphisms of finite presentation becomes finite étale after passing to such a stratification. In Section 5 we prove Theorem D and in Section 6 we prove a more general devissage result for quasi-finite and flat morphisms. In Section 7 we show that every stack with quasi-finite diagonal has a quasi-finite presentation and give an étale-local structure theorem of such stacks.

In Appendix A we state our conventions for algebraic stacks and give some technical results on separation axioms for stacks. In Appendix B we show that points on quasi-separated stacks are algebraic. In Appendix C we give two lemmas for algebraic spaces and in Appendices D and E we give a short introduction to 2-sheaves on stacks.

A morphism of stacks $f: X \rightarrow Y$ is *étale* if and only if f is locally of finite presentation, flat and has étale diagonal, cf. [Ryd09a, App. B].

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1. DESCENT FOR ÉTALE NEIGHBORHOODS

In this section we prove Theorems A and B. Recall that if X is an algebraic stack and $X = U_1 \cup U_2$ is an open covering, so that $U_2 \rightarrow X$ is an open neighborhood of $X \setminus U_1$, then a quasi-coherent sheaf on X can be described as quasi-coherent sheaves $\mathcal{F}_1 \in \mathbf{QCoh}(U_1)$ and $\mathcal{F}_2 \in \mathbf{QCoh}(U_2)$ together with an isomorphism $\mathcal{F}_1|_{U_1 \cap U_2} \rightarrow \mathcal{F}_2|_{U_1 \cap U_2}$.

The following notation will be fixed throughout this section.

Notation (1.1). Let X be an algebraic stack and let $Z \hookrightarrow X$ be a closed substack. Let $f: X' \rightarrow X$ be an étale neighborhood of $|Z|$, let $U = X \setminus Z$ and let $U' = X' \setminus Z = f^{-1}(U)$. Let $\mathbf{F}: \mathbf{Stack}_{\text{ét}/X}^{\text{op}} \rightarrow \mathbf{Cat}$ be a 2-presheaf, i.e., a (pseudo) 2-functor (cf. Appendix D). We have pull-back functors

$$(f|_U)^*: \mathbf{F}(U) \rightarrow \mathbf{F}(U'),$$

$$|_{U'}: \mathbf{F}(X') \rightarrow \mathbf{F}(U'),$$

we can form the 2-fiber product

$$\mathbf{F}(U) \times_{\mathbf{F}(U')} \mathbf{F}(X'),$$

and there is an induced functor

$$(|_U, f^*): \mathbf{F}(X) \rightarrow \mathbf{F}(U) \times_{\mathbf{F}(U')} \mathbf{F}(X')$$

which is unique up to unique natural isomorphism.

Under the assumption that \mathbf{F} is a 2-sheaf in the étale topology, we will show that the functor $(|_U, f^*)$ is an equivalence of categories. This is Theorem A. Examples of 2-sheaves include \mathbf{QCoh} and $\mathbf{Hom}(-, Y)$, cf. Appendix E.

Example (1.2). Let $\mathbf{F} = \mathbf{QCoh}$ be the 2-functor of quasi-coherent sheaves. Then

$$\mathbf{QCoh}(U) \times_{\mathbf{QCoh}(U')} \mathbf{QCoh}(X')$$

is the category of quasi-coherent sheaves on U with a specified extension to X' . More formally, the objects are triples $(\mathcal{F}_U, \theta, \mathcal{F}')$ where $\mathcal{F}_U \in \mathbf{QCoh}(U)$, $\mathcal{F}' \in \mathbf{QCoh}(X')$ and $\theta: (f|_U)^* \mathcal{F}_U \rightarrow \mathcal{F}'|_{U'}$ is an isomorphism. The morphisms are pairs $(\varphi_U, \varphi'): (\mathcal{F}_U, \theta, \mathcal{F}') \rightarrow (\mathcal{G}_U, \psi, \mathcal{G}')$ where $\varphi_U: \mathcal{F}_U \rightarrow \mathcal{G}_U$ and $\varphi': \mathcal{F}' \rightarrow \mathcal{G}'$ are homomorphisms such that $\varphi'|_{U'} \circ \theta = \psi \circ (f|_U)^* \varphi_U$.

Proof of Theorem A. Let $\pi_1, \pi_2: X' \times_X X' \rightarrow X'$ be the two projections, let $\Delta_{X'/X}: X' \rightarrow X' \times_X X'$ be the diagonal and let $\pi: X' \times_X X' \rightarrow X$ denote the structure morphism. The key observation is that the assumptions on f imply that

$$h = (j'' \amalg \Delta_{X'/X}): (U' \times_U U') \amalg X' \rightarrow X' \times_X X'$$

is étale, representable and surjective. Here $j'': U' \times_U U' \rightarrow X' \times_X X'$ denotes the canonical open immersion.

We first show that the functor $(|_U, f^*)$ is fully faithful. Let $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$ be two objects. Replacing X' with $X' \amalg U$ we can assume that f is surjective. As f is a morphism of descent, the sequence

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}(\mathcal{F}', \mathcal{G}') \xrightarrow[\pi_2^*]{\pi_1^*} \mathrm{Hom}(\mathcal{F}'', \mathcal{G}'')$$

is exact where $\mathcal{F}' = f^*\mathcal{F}$, $\mathcal{F}'' = \pi^*\mathcal{F}$ etc. As h is étale and surjective, the map

$$\mathrm{Hom}(\mathcal{F}'', \mathcal{G}'') \xrightarrow{h^*} \mathrm{Hom}(h^*\mathcal{F}'', h^*\mathcal{G}'')$$

is injective. Given compatible morphisms $\varphi_U: \mathcal{F}|_U \rightarrow \mathcal{G}|_U$ and $\varphi': \mathcal{F}' \rightarrow \mathcal{G}'$, we have that $h^*\pi_1^*\varphi' = h^*\pi_2^*\varphi'$ since both morphisms coincide with $(\pi|_U)^*\varphi_U$ on the first component $U' \times_U U'$ and with φ' on the second component X' . Thus, by descent, there is a unique morphism $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ such that $\varphi' = f^*\varphi$ and $\varphi_U = \varphi|_U$.

Next, we show that the functor is essentially surjective. Let $\mathcal{F}_U \in \mathbf{F}(U)$ and $\mathcal{F}' \in \mathbf{F}(X')$ be objects together with an isomorphism $\theta: (f|_U)^*\mathcal{F}_U \rightarrow \mathcal{F}'|_{U'}$. The isomorphism θ provides $\mathcal{F}'|_{U'}$ with a descent datum, i.e., an isomorphism $\psi_{U'}: (\pi_1^*\mathcal{F}')|_{U' \times_U U'} \rightarrow (\pi_2^*\mathcal{F}')|_{U' \times_U U'}$ satisfying the cocycle condition over $U' \times_U U' \times_U U'$. The cocycle condition implies that $(\Delta_{U'/U})^*\psi_{U'}$ is the identity on $\mathcal{F}'|_{U'}$.

As we have seen, the functor $h^* = (j'^*, (\Delta_{X'/X})^*): \mathbf{F}(X' \times_X X') \rightarrow \mathbf{F}(U' \times_U U') \times_{\mathbf{F}(U')} \mathbf{F}(X')$ is fully faithful so the isomorphism $(\psi_{U'}, \mathrm{id}_{X'})$ descends to a unique isomorphism $\psi: \pi_1^*\mathcal{F}' \rightarrow \pi_2^*\mathcal{F}'$ such that $\psi|_{U' \times_U U'} = \psi_{U'}$ and $(\Delta_{X'/X})^*\psi = \mathrm{id}_{X'}$. Finally, ψ satisfies the cocycle condition since $X' \times_X X' \times_X X'$ has an étale cover consisting of the open substack $U' \times_U U' \times_U U'$ and the diagonal $X' \rightarrow X' \times_X X' \times_X X'$. By effective descent, we obtain an object $\mathcal{F} \in \mathbf{F}(X)$ which restricts to \mathcal{F}_U and \mathcal{F}' . \square

Remark (1.3). Let $Z_1 \rightarrow X$ and $Z_2 \rightarrow X$ be morphisms of algebraic stacks. Theorem A applied to the 2-sheaf $\mathbf{F} = \mathbf{Hom}_X(Z_1 \times_X -, Z_2)$ shows that a morphism $\varphi_U: Z_1|_U \rightarrow Z_2|_U$ which extends to a morphism $\varphi': Z'_1 \rightarrow Z'_2$ descends to a morphism $\varphi: Z_1 \rightarrow Z_2$ which is unique up to unique 2-isomorphism. It can also be shown that a stack Z_U over U which extends to a stack Z' over X' glues to a stack Z over X which is unique up to unique 2-isomorphism, cf. Corollary (3.3).

A natural way to formalize these two results is to let \mathbf{F} be the “fibered 2-category of stacks” so that $\mathbf{F}(X)$ is the 2-category of stacks over X . The results are then equivalent to the statement that the 2-functor $(|_U, f^*): \mathbf{F}(X) \rightarrow \mathbf{F}(U) \times_{\mathbf{F}(U')} \mathbf{F}(X')$ is a 2-equivalence of 2-categories. The proof is straightforward except that one has to deal with unpleasant objects such as functors of tricategories or fibered 2-categories of stacks. The canonical descent datum in this setting consists of a 1-isomorphism over $(X'/X)^2 = X' \times_X X'$ and a 2-isomorphism over $(X'/X)^3$ satisfying a cocycle condition over $(X'/X)^4$. All these technical issues can be completely avoided using Theorem C as is done in Corollary (3.3).

Proof of Theorem B. Let Z be an algebraic stack and let

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ f|_U \downarrow & \varphi \nearrow & \downarrow g' \\ U & \xrightarrow{g_U} & Z \end{array}$$

be 2-commutative. We have to show that there is a morphism $g: X \rightarrow Z$ and a 2-commutative diagram

$$(1.3.1) \quad \begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & \nearrow \tau & \downarrow f \\ U & \xrightarrow{j} & X \\ & \searrow g_U & \downarrow \eta_U \\ & & Z \end{array} \quad \begin{array}{c} \nearrow g' \\ \nearrow \eta' \\ \nearrow g \end{array}$$

such that the pasting of the diagram is φ .

By Theorem (E.2), $\mathbf{F} = \mathbf{Hom}(-, Z): \mathbf{Stack} \rightarrow \mathbf{Cat}$ is a 2-sheaf. By Theorem A the object

$$(g_U, \varphi, g') \in \mathbf{F}(U) \times_{\mathbf{F}(U')} \mathbf{F}(X')$$

descends to an object $g \in \mathbf{F}(X) = \mathbf{Hom}(X, Z)$ together with a 2-morphism $\eta: (g \circ j, \tau, g \circ f) \Rightarrow (g_U, \varphi, g')$, i.e., we have two 2-morphisms $\eta_U: g \circ j \Rightarrow g_U$ and $\eta': g \circ f \Rightarrow g'$ such that the pasting of diagram (1.3.1) is φ .

Moreover, as $(|_U, f^*)$ is fully faithful, any two solutions (g, η) and $(\tilde{g}, \tilde{\eta})$ are uniquely 2-isomorphic. Specifically, there is a unique 2-isomorphism $\psi: g \Rightarrow \tilde{g}$ such that

$$\begin{array}{ccc} U & \longrightarrow & X \\ & \searrow & \downarrow \eta_U \\ & & Z \end{array} \quad \begin{array}{c} \xrightarrow{g} \\ \Downarrow \psi \\ \xrightarrow{\tilde{g}} \end{array} \quad \begin{array}{c} \nearrow \\ \nearrow \eta' \\ \nearrow g \end{array} \quad = \quad \begin{array}{ccc} U & \longrightarrow & X \\ & \searrow & \downarrow \eta_U \\ & & Z \end{array} \quad \begin{array}{c} \xrightarrow{g} \\ \Downarrow \eta_U \\ \xrightarrow{g} \end{array}$$

and such that the analogous identity involving η' and $\tilde{\eta}'$ holds. \square

Remark (1.4). The special case of Theorem A when X' and X are spectra of DVRs has been proved by Bosch, Lütkebohmert and Raynaud [BLR90, 6.2, C]. They also prove the more general result where $X' \rightarrow X$ is a *flat neighborhood* of DVRs [BLR90, 6.2, D], e.g., if X' is the completion of X .

Theorems A and B immediately generalize to *flat and finitely presented* neighborhoods. Indeed, if $X' \rightarrow X$ is flat and finitely presented, then $X' \rightarrow X$ is étale in an open neighborhood of $Z \hookrightarrow X'$. On the other hand, the straight-forward generalization of Theorem C to the flat and finitely presented case does not hold.

A perhaps more interesting case is when $X' \rightarrow X$ is flat and quasi-compact, e.g., the completion along a closed subscheme $Z \hookrightarrow X$. It is likely that Theorems A and B also hold in this generality, at least if we restrict the discussion to a suitable category of stacks so that all objects are stacks in the fpqc-topology.

2. ÉTALE NEIGHBORHOODS

Let X be an algebraic stack, let $Z \subseteq |X|$ be a closed subset and let $U = X \setminus Z$. Let $f: X' \rightarrow X$ be an étale neighborhood of Z . In this section we study how properties of X and f are related to properties of U , X' and

$f|_U$, e.g., f is representable if and only if $f|_U$ is representable. We begin by showing that the notion of being an étale neighborhood of Z is set-theoretic and does not depend on the choice of a stack-structure on Z .

Lemma (2.1). *Let $f: X' \rightarrow X$ be an étale morphism of algebraic stacks and let $Z \subseteq |X|$ be a closed subset. The following are equivalent*

- (i) *For every morphism $g: T \rightarrow X$ such that $g(T) \subseteq Z$, the projection $X' \times_X T \rightarrow T$ is an isomorphism.*
- (ii) *The projection $X' \times_X Z_{\text{red}} \rightarrow Z_{\text{red}}$ is an isomorphism, i.e., f is an étale neighborhood of Z .*
- (iii) *For every field k and point $x: \text{Spec}(k) \rightarrow X$ in Z , the fiber $X'_x \rightarrow \text{Spec}(k)$ is an isomorphism.*

Proof. Clearly (i) \implies (ii) \implies (iii). That (iii) \implies (i) follows immediately from the following two facts. A morphism which is locally of finite type and such that every fiber is an isomorphism is a surjective monomorphism [EGA_{IV}, Prop. 17.2.6]. A surjective étale monomorphism is an isomorphism [EGA_{IV}, Thm. 17.9.1]. \square

(2.2) Inertia stacks — Let $f: X \rightarrow Y$ be a morphism of stacks. Then there is an induced morphism of inertia stacks $I_f: I_X \rightarrow I_Y$. The morphism I_f is a composition $I_X \rightarrow I_Y \times_Y X \rightarrow I_Y$ where the first morphism is a pull-back of Δ_f and the second morphism is a pull-back of f . In particular, if f is étale (resp. an open immersion) then so is I_f .

(2.3) Given a cartesian diagram of stacks

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ f_U \downarrow & \square & \downarrow f \\ U & \xrightarrow{j} & X \end{array}$$

we have the following cartesian diagram of stacks

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ \Delta_{f_U} \downarrow & \square & \downarrow \Delta_f \\ U' \times_U U' & \xrightarrow{j' \times j'} & X' \times_X X' \end{array} \quad \text{and} \quad \begin{array}{ccc} I_{U'} & \xrightarrow{I_{j'}} & I_{X'} \\ I_{f_U} \downarrow & \square & \downarrow I_f \\ I_U & \xrightarrow{I_j} & I_X. \end{array}$$

Proposition (2.4). *Let $j: U \rightarrow X$ be an open immersion of stacks and let $f: X' \rightarrow X$ be an étale neighborhood of $Z = X \setminus U$. Let $j': U' \rightarrow X'$ be the pull-back of j along f . Then*

- (i) *$f \amalg j: X' \amalg U \rightarrow X$ is étale and surjective.*
- (ii) *Δ_f is an étale neighborhood of $X' \times_X X' \setminus U' \times_U U'$.*
- (iii) *I_f is an étale neighborhood of $I_X \setminus I_U$.*
- (iv) *$I_{X'} \rightarrow I_X \times_X X'$ is an étale neighborhood of $I_X \times_X X' \setminus I_U \times_U U'$.*
- (v) *If j' is quasi-compact then so is j .*
- (vi) *If $f|_U$ is an open immersion (resp. a quasi-compact open immersion, resp. an open and closed immersion, resp. an isomorphism, resp. surjective) then so is f .*

- (vii) If $f|_U$ is representable (resp. representable and quasi-separated, resp. representable and separated) then so is f .
- (viii) If j' is quasi-compact and $f|_U$ is quasi-compact (resp. quasi-separated, resp. of finite presentation) then f is quasi-compact (resp. quasi-separated, resp. of finite presentation).
- (ix) If U and X' are algebraic spaces, then so is X .
- (x) If j' is quasi-compact and U and X' are quasi-separated, then X is quasi-separated.
- (xi) If $f|_U$ is representable and U and X' have separated (resp. locally separated) diagonals, then X has separated (resp. locally separated) diagonal.

Proof. (i)–(iv) are obvious.

(v): If j' is quasi-compact then so is j since the pull-back of j along the étale surjective morphism $f \amalg j$ is $j' \amalg \text{id}_U$.

(vi): If $f|_U$ is an open immersion, then f is a monomorphism and hence an open immersion. If in addition $f|_U$ is quasi-compact (resp. closed), then so is f since the pull-back of f along the open covering $X' \amalg U \rightarrow X$ is quasi-compact (resp. closed). If $f|_U$ is surjective, then so is f .

(vii): Apply (vi) to the diagonal $\Delta_{X'/X}$.

(viii): Assume that $f|_U$ and j' are quasi-compact. The pull-back of f along $f \amalg j$ is $\pi_2 \amalg f|_U$ where $\pi_2: X' \times_X X' \rightarrow X'$ is the second projection. By assumption $f|_U$ is quasi-compact and π_2 is quasi-compact as the composition $\pi_2 \circ (\Delta_{X'/X} \amalg (j' \times j')): X' \amalg (U' \times_U U') \rightarrow X'$ is quasi-compact. Thus, f is quasi-compact.

Similarly, if j' and the diagonal of $f|_U$ are quasi-compact we apply the previous argument to the étale neighborhood Δ_f of $X' \times_X X' \setminus U' \times_U U'$ and conclude that Δ_f is quasi-compact. In particular, if $f|_U$ is quasi-separated (i.e., if its diagonal and the diagonal of the diagonal are quasi-compact) it follows that f is quasi-separated.

(ix): Assume that U and X' are algebraic spaces so that $I_U = I_X \times_X U \rightarrow U$ and $I_{X'} \rightarrow X'$ are isomorphisms. To show that X is an algebraic space it is thus enough to show that $I_X \times_X X' \rightarrow X'$ is an isomorphism. By (iv) we have an étale neighborhood as described by the diagram

$$\begin{array}{ccc} U' = I_{U'} & \longrightarrow & X' = I_{X'} \\ \downarrow & & \downarrow \\ U' = I_U \times_U U' & \longrightarrow & I_X \times_X X' \end{array}$$

and it follows that $I_X \times_X X' = X'$ by (vi).

(x): First note that $f|_U$ is quasi-separated so that f is quasi-separated by (viii). We have to prove that Δ_X is quasi-compact and quasi-separated. This is an étale-local question on $X \times X$ so it is enough to show that the pull-backs of Δ_X along $j \times \text{id}_X$, $\text{id}_X \times j$ and $f \times f$ are quasi-compact and quasi-separated. The first pull-back is $(\text{id}_U, j): U \rightarrow U \times X$ which is quasi-compact and quasi-separated since j is quasi-compact and quasi-separated and U is quasi-separated. The second pull-back is similar to the first one. The third pull-back is $X' \times_X X' \rightarrow X' \times X'$. Since f is quasi-separated,

$$\Delta_f \amalg j' \times j': X' \amalg U' \times_U U' \rightarrow X' \times_X X'$$

is quasi-compact, quasi-separated and surjective. It thus follows that $X' \times_X X' \rightarrow X' \times X'$ is quasi-compact and quasi-separated from Lemma (A.6) since $\Delta_{X'}$, Δ_U and j' are quasi-compact and quasi-separated.

(xi): Assume that $f|_U$ is representable so that f is representable by (vii). Further assume that Δ_U and $\Delta_{X'}$ are separated (resp. locally separated) so that the unit sections $U \rightarrow I_U$ and $X' \rightarrow I_{X'}$ are closed immersions (resp. immersions). To see that Δ_X is separated (resp. locally separated), we have to show that $X \rightarrow I_X$ is a closed immersion (resp. an immersion). As $X' \amalg U \rightarrow X$ is étale and surjective and $U \rightarrow I_U = I_X \times_X U$ is a closed immersion (resp. an immersion), it is enough to show that $X' \rightarrow I_X \times_X X'$ is a closed immersion (resp. an immersion). As f is representable, we have that $I_X \times_X X'$ is the union of two open subsets $I_U \times_U U'$ and $I_{X'}$. The restrictions of $X' \rightarrow I_X \times_X X'$ to these open subsets are $U' \rightarrow I_U \times_U U'$ and $X' \rightarrow I_{X'}$ which both are closed immersions (resp. immersions). \square

Examples (2.5). We give some examples showing that X and f can be rather “bad” even if $f|_U$, X' and U are “nice”.

- (i) ($f|_U$ finite but f not proper) Let $U \subset X$ be an open non-closed subset, let $X' = U \amalg X$ and let $f: X' \rightarrow X$ be the natural morphism. Then f is a non-proper étale neighborhood of $X \setminus U$ and $f|_U$ is finite.
- (ii) ($f|_U$ proper but f not separated) Let $U \subseteq X$ be an open non-closed subset and let $G \rightarrow X$ be the group scheme $G = X \amalg U \subseteq X \times \mathbb{Z}/2\mathbb{Z}$. Let $X' = BG = [X/G]$ so that $U' = U \times B(\mathbb{Z}/2\mathbb{Z})$. Then $f: X' \rightarrow X$ is an étale non-separated neighborhood of $X \setminus U$ and $f|_U$ is proper (a trivial étale $\mathbb{Z}/2\mathbb{Z}$ -gerbe).
- (iii) ($f|_U$ proper, U and X' separated but Δ_X not separated) Let $U \subseteq Y$ be an open non-closed subset. Let $G' = Y \times \mathbb{Z}/2\mathbb{Z}$ be the constant group scheme, let $H = Y \amalg U \subseteq G'$ be the induced subgroup and let $G = Y \amalg_U Y = G'/H$ so that G is a non-separated group scheme. Let $X = [Y/G]$ and $X' = [Y/G']$ where both group actions are trivial. Then $f: X' \rightarrow X$ is an étale neighborhood of $X \setminus U$ such that U and X' are separated and $f|_U$ is proper.
- (iv) ($f|_U$ proper, U and X' separated but Δ_X not locally separated) Let p be a prime, let $G' = \mu_{p,\mathbb{Z}}$, let $H = \mu_{p,\mathbb{Z}[1/p]} \amalg_{\mathrm{Spec}(\mathbb{Z}[1/p])} \mathrm{Spec}(\mathbb{Z}) \subseteq G'$ and let $G = G'/H$ so that G is not locally separated. Indeed, $G \rightarrow \mathrm{Spec}(\mathbb{Z})$ is a flat birational universal homeomorphism but not an isomorphism. Then let $X = [\mathrm{Spec}(\mathbb{Z})/G]$, $X' = [\mathrm{Spec}(\mathbb{Z})/G']$ and $U = \mathrm{Spec}(\mathbb{Z}[1/p])$.
- (v) ($f|_U$ finite, U and X' separated schemes but X not separated) Let $U = \mathbb{A}^1$ be the affine line, let $U' = U \amalg U$ and let $X' = \mathbb{P}^1 \amalg \mathbb{P}^1$. Then X is a non-separated “projective” line.
- (vi) ($f|_U$ finite, U and X' separated schemes but X not locally separated) Let $U = \mathbb{A}^1$ be the affine line, let $U' = U \amalg U$ and let $X' = \mathbb{P}^1 \amalg_{\infty} \mathbb{P}^1$ be two secant lines. Then X is a standard example of a non-locally separated algebraic space.

3. ÉTALE GLUINGS OF STACKS

In this section we will prove Theorem C on the existence of the push-out of an open immersion $j': U' \rightarrow X'$ and an étale morphism $f_U: U' \rightarrow U$.

If U , U' and X' are *algebraic spaces*, then it is rather straight-forward to construct the push-out X of j' and f_U . Indeed, by Theorem B we know a priori that $X' \times_X X'$ has to be the push-out of j' and $\Delta_{U'/U}$ and by assumption both these maps are open immersions so we can construct the algebraic space $R = X' \times_X X'$ as this push-out. The universal property of the push-out gives a morphism $R \rightarrow X' \times X'$ and it can be shown that this is an étale equivalence relation. The space X is then the quotient of this equivalence relation. Similarly, if U' and X' are algebraic spaces and U is an algebraic stack, then we can construct R as above and equip (R, X') with a groupoid structure although it is slightly tedious to verify that this is indeed a groupoid.

For arbitrary X' this procedure is not so straight-forward as the groupoid $R \rightrightarrows X'$ would be a groupoid in stacks (and even with non-representable morphisms if f_U is not representable!). The most natural approach is to first define X as a 2-stack and then show that X is equivalent to a 1-stack. To avoid the language of 2-stacks we will however instead do an explicit, albeit somewhat less natural, construction.

Proof of Theorem C. When f_U is a monomorphism (resp. representable, resp. arbitrary) then the diagonal $\Delta_{U'/U}$ is an isomorphism (resp. a monomorphism, resp. representable). We will assume that the theorem is true when f_U is an isomorphism (resp. a monomorphism, resp. representable) and show that the theorem is true when f_U is a monomorphism (resp. representable, resp. arbitrary). When f_U is an isomorphism, then $X = X'$ is the push-out.

We can thus assume that the push-out R' of $\Delta_{U'/U}$ and j' exists and fits into the bi-cartesian square

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ \Delta_{U'/U} \downarrow & & \downarrow \Delta \\ U' \times_U U' & \longrightarrow & R'. \end{array}$$

For $k = 1, 2$, the morphisms $j' \circ \pi_k: U' \times_U U' \rightarrow X'$ and $\text{id}_{X'}$ induce a morphism $q_k: R' \rightarrow X'$. When the existence of the push-out X has been settled, then $R' = X' \times_X X'$ and under this identification Δ becomes the diagonal and q_k the projection onto the k^{th} factor.

Let $p: X'_1 \rightarrow X'$ be a smooth presentation and let $X'_2 \rightrightarrows X'_1$ be the induced groupoid with quotient X' . Let $U'_2 \rightrightarrows U'_1$ be the pull-back of the groupoid along j' .

Consider the following category \mathbf{C} fibered over \mathbf{Sch} (we will eventually show that this is the push-out X). An object of \mathbf{C} over $T \in \mathbf{Sch}$ consists of

- (i) an open subset $T_\circ \subseteq T$,
- (ii) a groupoid $T'_2 \rightrightarrows T'_1$ over T ,
- (iii) a morphism $g_\circ: T_\circ \rightarrow U$,
- (iv) morphisms $g'_1: T'_1 \rightarrow X'_1$ and $g'_2: T'_2 \rightarrow X'_2$,

such that

(a) the diagrams

$$\begin{array}{ccc} T'_2 & \xrightarrow{g'_2} & X'_2 \\ s \downarrow & & \downarrow s \\ T'_1 & \xrightarrow{g'_1} & X'_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} T'_2 & \xrightarrow{g'_2} & X'_2 \\ t \downarrow & & \downarrow t \\ T'_1 & \xrightarrow{g'_1} & X'_1 \end{array}$$

are cartesian,

(b) the inverse images $T'_{1\circ} \subseteq T'_1$ and $T'_{2\circ} \subseteq T'_2$ of $T_\circ \subseteq T$ coincide with $(g'_1)^{-1}(U'_1)$ and $(g'_2)^{-1}(U'_2)$,

(c) the diagram

$$\begin{array}{ccc} T'_{1\circ} & \longrightarrow & U'_1 \\ \downarrow & & \downarrow \\ T_\circ & \longrightarrow & U \end{array}$$

is cartesian,

(d) the stack quotient $T' = [T'_2 \rightrightarrows T'_1]$ is an étale neighborhood of $T \setminus T_\circ$ in T .

For an object $(T_\circ \subseteq T, T'_\bullet \rightarrow T, g_\circ, g'_\bullet)$ we thus obtain a cartesian diagram

$$\begin{array}{ccccc} U' & \xrightarrow{j'} & X' & & \\ \downarrow f_U & \swarrow & \downarrow g' & \square & \\ U & & T'_\circ & \longrightarrow & T' \\ & \nwarrow g_\circ & \downarrow & \square & \downarrow \\ & & T_\circ & \longrightarrow & T \end{array}$$

where the bottom-right square is bi-cartesian by Theorem B.

A morphism in \mathbf{C} over $S \rightarrow T$ is a commutative diagram

$$\begin{array}{ccccccc} S'_2 & \rightrightarrows & S'_1 & \longrightarrow & S & \longleftarrow & S_\circ \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T'_2 & \rightrightarrows & T'_1 & \longrightarrow & T & \longleftarrow & T_\circ \rightleftarrows \\ \downarrow & & \downarrow & & & & \downarrow \\ X'_2 & \rightrightarrows & X'_1 & & & & U \end{array}$$

such that all natural squares are cartesian.

By étale descent of algebraic spaces, the category \mathbf{C} is a stack.

Let $j: U \rightarrow \mathbf{C}$ be the morphism taking a morphism $h: T \rightarrow U$ to the object of $\mathbf{C}(T)$ given by

- $T_\circ = T$ and $g_\circ = h: T_\circ = T \rightarrow U$,
- $T'_i = U'_i \times_U T$ and $g'_i = j'_i \circ \pi_1: T'_i \rightarrow U'_i \rightarrow X'_i$ for $i = 1, 2$.

The pull-back of an object $(T_\circ \subseteq T, T'_\bullet \rightarrow T, g_\circ, g'_\bullet)$ along j is $(T_\circ \subseteq T_\circ, T'_{\bullet\circ} \rightarrow T_\circ, g_\circ, g'_{\bullet\circ})$ and hence j is an open immersion.

Let $f: X' \rightarrow \mathbf{C}$ be the morphism taking a morphism $h: T \rightarrow X'$ to the object of $\mathbf{C}(T)$ given by

- $T_\circ = h^{-1}(U')$ and $g_\circ = f_U \circ h|_{U'}: T_\circ \rightarrow U$,
- $T'_i = T \times_{X', q_1} R' \times_{q_2, X'} X'_i$ for $i = 1, 2$, and the morphisms $g'_i = \pi_3: T'_i \rightarrow X'_i$ for $i = 1, 2$.

In particular, we have a groupoid $T'_1 \rightrightarrows T'_2$ with quotient $T' = T \times_{X', q_1} R'$ and the induced map $g': T' \rightarrow X'$ is $q_2 \circ \pi_2$. We note that there is a section $s = (\text{id}_T, \Delta \circ h): T \rightarrow T'$ and a 2-morphism $g' \circ s \Rightarrow h \circ \pi_1 \circ s = h$.

Let $(S_\circ \subseteq S, S'_\bullet \rightarrow S, g_\circ, g'_\bullet)$ be an object of $\mathbf{C}(S)$. We will now show that the square

$$(3.0.1) \quad \begin{array}{ccc} S' & \xrightarrow{g'} & X' \\ \downarrow & & \downarrow f \\ S & \longrightarrow & \mathbf{C} \end{array}$$

is 2-cartesian.

The first step is to show that it is 2-commutative. Let T be a scheme and let $T \rightarrow S'$ be a morphism. Let $T_\circ = T \times_{S'} S'_\circ$ so that $T \times_{X'} U' = T_\circ = T \times_S S_\circ$. The composition $T \rightarrow S' \rightarrow S \rightarrow \mathbf{C}$ corresponds to the object

$$(3.0.2) \quad \begin{array}{ccccc} T'_\bullet & \longrightarrow & T & \longleftarrow & T_\circ \\ \downarrow & & & & \downarrow \\ X'_\bullet & & & & U \end{array}$$

where $T'_i = S'_i \times_S T$. The second composition $T \rightarrow S' \rightarrow X' \rightarrow \mathbf{C}$ corresponds to the object

$$(3.0.3) \quad \begin{array}{ccccc} T \times_{X', q_1} R' \times_{q_2, X'} X'_\bullet & \longrightarrow & T & \longleftarrow & T_\circ \\ \downarrow & & & & \downarrow \\ X'_\bullet & & & & U. \end{array}$$

We have the following bi-cartesian squares

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \Delta_{U'/U} \downarrow & & \downarrow \Delta \\ U' \times_U U' & \longrightarrow & R' \end{array} \quad \text{and} \quad \begin{array}{ccc} S'_\circ & \longrightarrow & S' \\ \Delta_{S'_\circ/S_\circ} \downarrow & & \downarrow \Delta_{S'/S} \\ S'_\circ \times_{S_\circ} S'_\circ & \longrightarrow & S' \times_S S'. \end{array}$$

The pull-back of these squares along $\pi_2: T \times_{X', q_1} R' \rightarrow R'$ and $T' = T \times_S S' \rightarrow S' \times_S S'$ respectively gives the bi-cartesian squares

$$\begin{array}{ccc} T_\circ & \longrightarrow & T \\ \downarrow & & \downarrow \\ T_\circ \times_U U' & \longrightarrow & T \times_{X', q_1} R' \end{array} \quad \text{and} \quad \begin{array}{ccc} T_\circ & \longrightarrow & T \\ \downarrow & & \downarrow \\ T'_\circ & \longrightarrow & T'. \end{array}$$

Note that $T'_\circ = T_\circ \times_U U'$ so that by the universal property of push-outs there is an isomorphism of stacks $T \times_{X', q_1} R' \cong T'$. It follows that there is

a 2-morphism

$$(T \rightarrow S' \rightarrow X' \rightarrow \mathbf{C}) \Rightarrow (T \rightarrow S' \rightarrow S \rightarrow \mathbf{C})$$

and hence the diagram (3.0.1) is 2-commutative.

To show that the diagram is 2-cartesian, let $k: T \rightarrow S$ and $h: T \rightarrow X'$ be morphisms with a given 2-morphism $(T \rightarrow X' \rightarrow \mathbf{C}) \Rightarrow (T \rightarrow S \rightarrow \mathbf{C})$. The morphisms $T \rightarrow S \rightarrow \mathbf{C}$ and $T \rightarrow X' \rightarrow \mathbf{C}$ correspond to objects as described in (3.0.2) and (3.0.3) and the 2-morphism gives a cartesian diagram

$$(3.0.4) \quad \begin{array}{ccccc} T \times_{X', q_1} R' \times_{q_2, X'} X'_{\bullet} & \longrightarrow & T & \longleftarrow & T_{\circ} \\ \downarrow \cong & & \parallel & & \parallel \\ T'_{\bullet} & \longrightarrow & T & \longleftarrow & T_{\circ} \\ \downarrow k'_{\bullet} & & \downarrow k & & \downarrow k_{\circ} \\ S'_{\bullet} & \longrightarrow & S & \longleftarrow & S_{\circ} \\ \downarrow g'_{\bullet} & & & & \downarrow g_{\circ} \\ X'_{\bullet} & & & & U. \end{array}$$

The section $s = (\text{id}_T, \Delta \circ h): T \rightarrow T \times_{X', q_1} R'$ induces a canonical section $s: T \rightarrow T'$ and a 2-morphism $g' \circ k' \circ s \Rightarrow h$ which fits into a 2-commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{h} & X' \\ \downarrow k' \circ s & \nearrow \text{2-morphism} & \downarrow f \\ S' & \xrightarrow{g'} & X' \\ \downarrow k & \searrow \text{2-morphism} & \downarrow f \\ S & \longrightarrow & \mathbf{C}. \end{array}$$

This shows that the square is 2-cartesian. Note that $X' \times_{\mathbf{C}} X' \cong R'$ as asserted in the beginning of the proof.

It follows that the morphism $f \amalg j: X' \amalg U \rightarrow \mathbf{C}$ is étale and surjective and that f is an étale neighborhood of $\mathbf{C} \setminus U$. Indeed, the pull-back of $f \amalg j$ along $S \rightarrow \mathbf{C}$ is $S' \amalg S_{\circ} \rightarrow S$ which is étale and surjective and $S' \rightarrow S$ is an étale neighborhood. In particular, \mathbf{C} admits a smooth presentation and is hence algebraic.

Finally we deduce from Theorem B that \mathbf{C} is the push-out of f and j' . As the pull-back of an étale neighborhood is an étale neighborhood, the push-out commutes with arbitrary base change. The remaining properties listed in Theorem C is part of Proposition (2.4). \square

Remark (3.1). If j' is quasi-compact and U and X' are quasi-separated, then $X = \mathbf{C}$ is *a posteriori* a quasi-separated stack. The reader who does not want to introduce a priori non quasi-separated stacks in the proof can verify directly that when j' is quasi-compact, then

$$(U \amalg X') \times_{\mathbf{C}} (U \amalg X') \rightarrow (U \amalg X') \times (U \amalg X')$$

is indeed representable, quasi-compact and quasi-separated so that \mathbf{C} is a quasi-separated stack.

Proposition (3.2). *Given a cartesian diagram of algebraic stacks*

$$\begin{array}{ccccc} U & \xleftarrow{f_U} & U' & \xrightarrow{j'} & X' \\ \downarrow & & \downarrow & & \downarrow \\ V & \xleftarrow{g_V} & V' & \xrightarrow{k'} & Y' \end{array}$$

such that j' and k' are open immersions and f_U and g_V are étale, let $X = X' \amalg_{U'} U$ and $Y = Y' \amalg_{V'} V$ denote the push-outs. Then every face of the induced cube

$$\begin{array}{ccccc} & & U' & \xrightarrow{j'} & X' \\ & \searrow & \downarrow f_U & \searrow f & \downarrow \\ & & V' & \xrightarrow{k'} & Y' \\ & \swarrow & \downarrow f_U & \swarrow f & \downarrow \\ U & \xrightarrow{j} & X & \xrightarrow{g} & Y \\ & \swarrow & \downarrow g_V & \swarrow g & \downarrow \\ & & V & \xrightarrow{k} & Y \end{array}$$

is cartesian.

Proof. We may replace Y , Y' , V and V' by their pull-backs along $X \rightarrow Y$ and assume that $X = Y$. Since f , g , j and k are étale, it is enough to verify that the cube is cartesian over points of Y . We can thus assume that $X = Y = \text{Spec}(k)$. But then either $V = Y$ which implies that the top and the bottom square are trivial and $f = g = f_U = g_V$, or $V = \emptyset$ which implies that $U = U' = V = V' = \emptyset$ and $X = X' = Y = Y'$. Thus in either case, we have that $U = V$ and $X' = Y'$. \square

Corollary (3.3). *Let $j: U \rightarrow X$ be an open immersion of algebraic stacks and let $f: X' \rightarrow X$ be an étale neighborhood of $X \setminus U$. Let $U' = f^{-1}(U)$. Let $Z_U \rightarrow U$ and $Z' \rightarrow X'$ be morphisms of stacks and let $Z'|_{U'} \cong Z_U \times_U U'$ be an isomorphism. Then there is a stack $Z \rightarrow X$, unique up to unique 2-morphism, and morphisms $Z' \rightarrow Z$ and $Z_U \rightarrow Z$ such that every face of the cube*

$$\begin{array}{ccccc} Z'|_{U'} & \xrightarrow{\quad} & Z' & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & & U' & \xrightarrow{j'} & X' \\ \downarrow & & \downarrow & & \downarrow \\ Z_U & \xrightarrow{\quad} & Z & \xrightarrow{f} & X \\ & \swarrow & \downarrow f|_U & \swarrow f & \\ & & U & \xrightarrow{j} & X \end{array}$$

is cartesian.

Proof. Any stack $Z \rightarrow X$ satisfying the condition of the Corollary is a push-out of $Z'|_{U'} \rightarrow Z'$ and $Z'|_{U'} \rightarrow Z_U$ by Theorem B. By Theorem C, the push-out Z exists and the cube is cartesian by Proposition (3.2). \square

4. CONSTRUCTIBLE SHEAVES

In this section, we show that given a constructible sheaf \mathcal{F} on a quasi-compact and quasi-separated stack X , there is a finite filtration of X in open quasi-compact substacks such that \mathcal{F} is locally constant on the induced stratification of X . We begin with a short review of constructible sheaves on stacks, cf. [SGA₄, Exp. IX, §2] and [LMB00, Ch. 18].

(4.1) Let X be a quasi-compact and quasi-separated stack. Recall that a subset $W \subseteq X$ is locally closed if W is the intersection of a closed and an open subset. A locally closed subset W is constructible if and only if W and its complement are quasi-compact, or equivalently, if and only if $W = U \setminus V$ where $V \subseteq U \subseteq X$ are open and quasi-compact.

(4.2) Let X be a quasi-compact and quasi-separated *scheme*. Let \mathcal{F} be a sheaf of sets on the small étale site of X . Recall that \mathcal{F} is *locally constant* if there exists a covering $\{U_i \rightarrow X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf for every U_i . The sheaf \mathcal{F} is *constructible* if there exists a finite stratification $X = \cup X_i$ into locally closed constructible subsets $X_i \subseteq X$ such that $\mathcal{F}|_{X_i}$ is locally constant and finite [SGA₄, Exp. IX, Prop. 2.4]. Note that the choice of scheme structure on the X_i 's is irrelevant since the étale sites of X_i and $(X_i)_{\text{red}}$ are equivalent.

Every constructible sheaf is represented by an algebraic space, étale and finitely presented over X [SGA₄, Exp. IX, Prop. 2.7]. In other words, there is a one-to-one correspondence between constructible sheaves on X and finitely presented étale morphisms $X' \rightarrow X$ of algebraic spaces given by taking X' to the corresponding sheaf of sections. Note that X' is a scheme if $X' \rightarrow X$ is separated [Knu71, Cor. 6.17]. A constructible sheaf is locally constant if and only if it is represented by a finite étale morphism.

(4.3) Let X be a quasi-compact and quasi-separated *stack* and let $\pi: V \rightarrow X$ be a presentation such that V is a quasi-compact and quasi-separated scheme (e.g., an affine scheme). Since π is open, surjective and quasi-compact, it follows that a subset $W \subseteq X$ is locally closed (resp. locally closed and constructible) if and only if $\pi^{-1}(W)$ is so. By definition, a sheaf of sets \mathcal{F} on the lisse-étale site of X , is locally constant (resp. constructible) if it is cartesian and $\pi^*\mathcal{F}$ is locally constant (resp. constructible) [LMB00, Déf. 18.1.4]. This definition does not depend on the choice of presentation. It follows, e.g., using local constructions as in [LMB00, Ch. 14], that the category of constructible sheaves on X is equivalent to the category $\mathbf{Stack}_{\text{repr,fp,ét}/X}$ of representable finitely presented and étale morphisms $X' \rightarrow X$.

Surprisingly, the following result (closely related to [LMB00, Prop. 18.1.7] and [SGA₄, Exp. IX, Prop. 2.5]) seems to be missing in the literature.

Proposition (4.4). *Let X be a quasi-compact and quasi-separated stack.*

- (i) Let \mathcal{F} be a lisse-étale sheaf of sets on X . Then \mathcal{F} is constructible if and only if there exists a finite filtration $\emptyset = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$ of open quasi-compact subsets such that $\mathcal{F}|_{X_i \setminus X_{i-1}}$ is locally constant of constant finite rank for every $i = 1, 2, \dots, n$.
- (ii) Let $f: X' \rightarrow X$ be a representable étale morphism. Then f is of finite presentation if and only if there exists a filtration of X as in (i) such that $f|_{X_i \setminus X_{i-1}}$ is finite and étale of constant rank for every $i = 1, 2, \dots, n$.

Proof. The two statements are equivalent. We will show the Proposition in the form (i). The condition is clearly sufficient. To prove necessity, let $\pi: V \rightarrow X$ be a presentation with V a quasi-compact and quasi-separated scheme. If there exists a filtration $\emptyset = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ of open quasi-compact subsets such that $\mathcal{F}|_{V_i \setminus V_{i-1}}$ is locally constant of constant finite rank, then the filtration of X given by $X_i = \pi(V_i)$ is sufficient. Replacing X with V , we can thus assume that X is a scheme.

By definition, there is then a stratification $X = \bigcup W_i$ into locally closed constructible subsets such that $\mathcal{F}|_{W_i}$ is locally constant and by refining the W_i 's, we can assume that the rank of $\mathcal{F}|_{W_i}$ is constant. Write $W_i = U_i \setminus V_i$ where $V_i \subseteq U_i \subseteq X$ are quasi-compact open subsets. Let T be the topology on X generated by all U_i 's and V_i 's. Then every element of T is a quasi-compact open subset and T is finite. Let $N = |T|$ be the number of open subsets. We will show the lemma by induction on N . Let $X_1 \in T$ be a non-empty minimal open subset. Then $\mathcal{F}|_{X_1}$ is locally constant of constant rank. By induction, the lemma holds for $Z = X \setminus X_1$ and $\mathcal{F}|_Z$ and we obtain a stratification on X by taking $X_i = X_1 \cup Z_{i-1}$. \square

Remark (4.5). In general, the stratification in Proposition (4.4) is not canonical. There are two important special cases though:

- (i) Let X be a quasi-compact and quasi-separated stack and let $f: X' \rightarrow X$ be a *separated* and quasi-compact étale morphism. Then the fiber rank of f is a constructible and lower semi-continuous function. Thus, there is a *canonical* finite filtration $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_{n+1} = \emptyset$ of X into open quasi-compact subsets, such that f is finite and étale of constant rank i over the locally closed constructible subset $X_i \setminus X_{i+1}$.

Similarly, if f is *universally closed* and finitely presented but not necessarily separated, the fiber rank of f is constructible and upper semi-continuous and we obtain a canonical filtration.

- (ii) Let X be a *noetherian* stack and let \mathcal{F} be a constructible sheaf. Let U be the maximal open subset such that $\mathcal{F}|_U$ is locally constant. We can then take X_1 as the open and closed subset of U with minimal fiber rank. Proceeding with $X \setminus X_1$ we obtain a canonical filtration by noetherian induction. If X is not noetherian, then this procedure would also give a finite filtration but the X_i 's would not necessarily be quasi-compact.

5. ÉTALE DEVISSAGE

In this section, we describe a devissage method for representable finitely presented étale morphisms. In the separated case, this devissage was used by Raynaud and Gruson to pass from algebraic spaces to schemes [RG71, §5.7]. We have taken some care to also include the non-separated case. This is motivated by Examples (5.6) and (5.7). The starting point is the existence of stratifications of finitely presented étale morphisms as in Proposition (4.4). The idea is to then use symmetric products to pass to étale neighborhoods.

Definition (5.1). Let $f: X' \rightarrow X$ be a representable morphism of stacks. We let $(X'/X)^d$ be the d^{th} fiber product of X' over X and we let the symmetric group \mathfrak{S}_d act on $(X'/X)^d$ by permuting the factors. Let $Z \hookrightarrow X$ be a closed subset of X such that $f|_Z$ is *separated* and let $Z' = f^{-1}(Z)$. Further let $\Delta_{Z'/Z}$ be the diagonal of $Z' \times_Z Z'$ as a closed subset of $X' \times_X X'$ and let $\Delta(Z)$ be the big diagonal of Z' in $(X'/X)^d$, i.e., the \mathfrak{S}_d -orbit of $\Delta_{Z'/Z} \times_X (X'/X)^{d-2}$. Then $\Delta(Z)$ is closed and we let $\text{SEC}_Z^d(X'/X) \subseteq (X'/X)^d$ be its complement. We let $\acute{\text{ET}}_Z^d(X'/X) = [\text{SEC}_Z^d(X'/X)/\mathfrak{S}_d]$ be the stack quotient.

Remark (5.2). The stack $\text{SEC}_Z^d(X'/X)$ parameterizes d sections of $f: X' \rightarrow X$ such that these sections are disjoint over Z . The stack $\acute{\text{ET}}_Z^d(X'/X)$ parameterizes finite étale morphisms $W \rightarrow X$ of rank d together with an X -morphism $W \rightarrow X'$ which is a closed immersion over $Z \hookrightarrow X$. If f is separated, then we can form $\acute{\text{ET}}^d(X'/X) := \acute{\text{ET}}_X^d(X'/X)$ which is the stack considered in [LMB00, 6.6]. If X'/X is étale of constant rank d , then $\acute{\text{ET}}^d(X'/X) \rightarrow X$ is an isomorphism.

Lemma (5.3). *Let $f: X' \rightarrow X$ be a representable étale surjective morphism of algebraic stacks, and let $Z \hookrightarrow X$ be a closed subset such that $f|_Z$ is finite of constant rank d .*

- (i) *The projections $\pi_1, \pi_2, \dots, \pi_d: \text{SEC}_Z^d(X'/X) \rightarrow X'$ are étale and surjective.*
- (ii) *$\acute{\text{ET}}_Z^d(X'/X) \rightarrow X$ is a surjective étale neighborhood of Z .*
- (iii) *If f is separated then $\acute{\text{ET}}^d(X'/X) \rightarrow X$ is a representable and separated étale neighborhood of Z .*

Proof. That π_i and $\acute{\text{ET}}_Z^d(X'/X) \rightarrow X$ are étale and surjective follows from the construction and $\acute{\text{ET}}_Z^d(X'/X)|_Z = \acute{\text{ET}}^d(f^{-1}(Z)/Z) \cong Z$. If f is separated, then \mathfrak{S}_d acts freely on $\text{SEC}_X^d(X'/X)$, relative to X , so that $\acute{\text{ET}}^d(X'/X) \rightarrow X$ is representable. \square

Proof of Theorem D. Let $f: X' \rightarrow X$ be representable, étale and surjective of finite presentation such that $X' \in \mathbf{D}$. Pick a filtration $\emptyset = X_0 \subset X_1 \subset X_2 \subset \dots \subset X_n = X$ of open quasi-compact subsets such that $f|_{X_i \setminus X_{i-1}}$ is finite and étale of constant rank d_i as in Proposition (4.4). Let $X'_i = f^{-1}(X_i)$. We will show that $X \in \mathbf{D}$ by induction. Thus, let $1 \leq i \leq n$ and assume that $X_{i-1} \in \mathbf{D}$.

Let $Z_i = X_i \setminus X_{i-1}$. If f is separated, let $\text{SEC}_i = \text{SEC}^{d_i}(X'_i/X_i)$ and $\acute{\text{ET}}_i = \acute{\text{ET}}^{d_i}(X'_i/X_i)$. If f is not separated, let $\text{SEC}_i = \text{SEC}_{Z_i}^{d_i}(X'_i/X_i)$ and $\acute{\text{ET}}_i = \acute{\text{ET}}_{Z_i}^{d_i}(X'_i/X_i)$. By Lemma (5.3) we have that $\acute{\text{ET}}_i \rightarrow X_i$ is an étale neighborhood of Z_i , that $\text{SEC}_i \rightarrow \acute{\text{ET}}_i$ is a finite étale surjective morphism of rank $d_i!$ and that there is a finitely presented étale morphism $\text{SEC}_i \rightarrow X'_i$. Moreover, if f is separated, then $\acute{\text{ET}}_i \rightarrow X_i$ is representable and separated.

By (D1) we have that $\text{SEC}_i \in \mathbf{D}$, by (D2) it follows that $\acute{\text{ET}}_i \in \mathbf{D}$ and by (D3) we have that $X_i \in \mathbf{D}$. It follows that $X \in \mathbf{D}$ by induction. \square

Remark (5.4). Condition (D1) of Theorem D states that \mathbf{D} is a sieve on \mathbf{E} , i.e., a presheaf $\mathbf{E}^{\text{op}} \rightarrow \{\emptyset, \{*\}\}$. Conditions (D2) and (D3) signify that this presheaf satisfies the sheaf condition with respect to finite and surjective étale morphisms and with respect to coverings of the form $(U \rightarrow X, X' \rightarrow X)$ where $U \rightarrow X$ is an open immersion and $X' \rightarrow X$ is an étale neighborhood of $X \setminus U$. The conclusion of Theorem D is that the presheaf satisfies the sheaf condition with respect to representable étale coverings. Theorem D can be generalized to arbitrary presheaves.

We end this section with some examples:

Example (5.5) (cf. [RG71, 5.7.6]). Let X be a quasi-compact and quasi-separated algebraic space. Then there exists an affine scheme X' and an étale presentation $f: X' \rightarrow X$. Since f is separated, the fiber rank of f is a lower semi-continuous constructible function. Thus, there is a *canonical* filtration $\emptyset = X_{n+1} \subseteq X_n \subseteq X_{n-1} \cdots \subseteq X_1 = X$ of quasi-compact open substacks X_i such that $f|_{X_d \setminus X_{d+1}}$ is finite of constant rank d . Let $\acute{\text{ET}}_d = \acute{\text{ET}}^d(X'_d/X_d)$ so that $\acute{\text{ET}}_d \rightarrow X_d$ is a representable, separated and surjective étale neighborhood of $X_d \setminus X_{d+1} \hookrightarrow X_d$. As $\acute{\text{ET}}_d$ is the quotient of the quasi-affine $\text{SEC}^d(X'_d/X_d)$ by a *free* group action, it is a quasi-affine scheme by Lemma (C.1). In particular, we have that $X_d \setminus X_{d+1}$ is a quasi-affine scheme.

Example (5.6) ([Ryd09a]). Let $f: X \rightarrow Y$ be an unramified morphism of algebraic stacks. Then there is a canonical factorization $X \hookrightarrow E_{X/Y} \rightarrow Y$ of f where $X \hookrightarrow E_{X/Y}$ is a closed immersion and $e: E_{X/Y} \rightarrow Y$ is étale. The étale morphism e is almost never separated but e is at least universally closed if f is finite. If f is representable and of finite presentation, then so is e . The devissage method can thus be extended to treat representable finitely presented unramified morphisms.

Example (5.7) ([LMB00, 6.8]). Let $f: X \rightarrow Y$ be a smooth and representable morphism. Then there is a canonical factorization $X \rightarrow \pi_0(X/Y) \rightarrow Y$ of f where $X \rightarrow \pi_0(X/Y)$ is smooth, representable and has geometrically connected fibers and $\pi: \pi_0(X/Y) \rightarrow Y$ is étale and representable. If f is *étale* then $f = \pi$ and π is separated if and only if f is separated. On the other hand, there are examples where f is smooth and separated but π is not separated.

6. QUASI-FINITE FLAT DEVISSAGE

Let S be an algebraic stack. We let $\mathbf{Stack}_{\text{fp}, \text{qff}/S}$ denote the category of quasi-finite and flat morphisms $X \rightarrow S$ of finite presentation and let $\mathbf{Stack}_{\text{repr}, \text{sep}, \text{fp}, \text{qff}/S}$ denote the subcategory of representable and separated morphisms. The main result of this section is the following theorem:

Theorem (6.1). *Let S be a quasi-compact and quasi-separated algebraic stack and let \mathbf{F} be either $\mathbf{Stack}_{\text{fp}, \text{qff}/S}$ or $\mathbf{Stack}_{\text{repr}, \text{sep}, \text{fp}, \text{qff}/S}$. Let $\mathbf{D} \subseteq \mathbf{F}$ be a full subcategory such that*

- (D1) *If $X \in \mathbf{D}$ and $(X' \rightarrow X) \in \mathbf{F}$ is étale then $X' \in \mathbf{D}$.*
- (D2) *If $X' \in \mathbf{D}$ and $(X' \rightarrow X) \in \mathbf{F}$ is finite and surjective, then $X \in \mathbf{D}$.*
- (D3) *If $j: U \rightarrow X$ and $f: X' \rightarrow X$ are morphisms in \mathbf{F} such that j is an open immersion and f is an étale neighborhood of $X \setminus U$, then $X \in \mathbf{D}$ if $U, X' \in \mathbf{D}$.*

Then if $(X' \rightarrow X) \in \mathbf{F}$ is representable, locally separated and surjective and $X' \in \mathbf{D}$, we have that $X \in \mathbf{D}$.

Note that the only difference between the conditions of Theorem D and Theorem (6.1) is that in the second condition $X' \rightarrow X$ is only required to be flat, not merely étale.

Remark (6.2). Let S be a quasi-compact and quasi-separated algebraic stack with quasi-finite diagonal. Then S admits a quasi-finite flat presentation $S' \rightarrow S$ by Theorem (7.1). If in addition S has *locally separated* diagonal, then we can arrange so that $S' \rightarrow S$ is locally separated. If $S' \in \mathbf{D}$ we can then apply Theorem (6.1) to deduce that $S \in \mathbf{D}$.

Theorem (6.1) is an immediate corollary of Theorem D and the following result about the étale-local structure of quasi-finite morphisms.

Theorem (6.3). *Let $f: X \rightarrow Y$ be a quasi-finite flat morphism of finite presentation between algebraic stacks. Then there exists a commutative diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & \circ & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

where the horizontal morphisms are étale and quasi-separated, where $X' \rightarrow X$ is surjective and where $f': X' \rightarrow Y'$ is finite, flat and of finite presentation. Moreover,

- (i) *If f is representable and separated, we can arrange so that the horizontal morphisms are representable, separated and of finite presentation.*
- (ii) *If f is representable and locally separated, we can arrange so that the horizontal morphisms are representable.*
- (iii) *If Y is quasi-compact and quasi-separated, we can arrange so that the horizontal morphisms are of finite presentation.*

Note that if Y is a quasi-compact Deligne–Mumford stack with quasi-compact and separated diagonal, then the result follows from the case where

Y is a scheme which is well known. The proof in the general case is much more subtle and is inspired by Keel and Mori's usage of Hilbert schemes [KM97, §4].

Proof. Let us first assume that f is representable and separated. The étale sheaf $f_! \left(\underline{\mathbb{Z}/2\mathbb{Z}}_X \right)$ is *constructible* and hence represented by a finitely presented étale morphism $Y' \rightarrow Y$. We let $X' \subset X \times_Y Y'$ be the support of the universal section. This is an open and closed subset so that $X' \rightarrow X$ is finitely presented and étale. By the definition of $f_!$ we have that f' is proper and hence finite. That $X' \rightarrow X$ is surjective can be checked after passing to fibers of f since $f_!$ commutes with arbitrary base change.

To see that $Y' \rightarrow Y$ is separated, we describe Y' as a Hilbert scheme. Let $\text{Hilb}_k^{\text{open}}(X/Y)$ be the open subscheme of the relative Hilbert scheme of k points on X/Y parameterizing open and closed subschemes. That is, for any scheme T and morphism $T \rightarrow Y$, the T -points of $\text{Hilb}_k^{\text{open}}(X/Y)$ are in bijection with open and closed subschemes $Z \hookrightarrow X \times_Y T$ such that $Z \rightarrow T$ is flat and finite of constant rank k . Then $Y' = \coprod_{k \geq 0} \text{Hilb}_k^{\text{open}}(X/Y)$ so that $Y' \rightarrow Y$ is separated.

We now drop the assumption that f is separated. It is still possible to define $f_! \left(\underline{\mathbb{Z}/2\mathbb{Z}}_X \right)$ but it does not carry a universal section. Instead, consider the Hilbert stack $\mathcal{H}_k^{\text{ét}}(X/Y)$ parameterizing flat families $Z \rightarrow T$ of constant rank k with an étale morphism $Z \rightarrow X \times_Y T$. The Hilbert stack $\mathcal{H}_k^{\text{ét}}(X/Y)$ is an open substack of the full Hilbert stack $\mathcal{H}_k(X/Y)$ which is known to be algebraic and of finite presentation [Ryd10, Thm. 4.4]. It is readily verified that $\mathcal{H}_k^{\text{ét}}(X/Y) \rightarrow Y$ is étale. We let $\mathcal{H} = \coprod_{k \geq 0} \mathcal{H}_k^{\text{ét}}(X/Y)$. In the general case we can then let $Y' = \mathcal{H}$ and let $X' \rightarrow Y'$ be the universal family. That $X' \rightarrow X$ is surjective can be checked on points of Y where it is obvious.

If f is representable and locally separated, then we let Y' be the largest open substack of \mathcal{H} such that $Y' \rightarrow Y$ is representable and let X' be the restriction of the universal family to Y' . It remains to verify that $X' \rightarrow X$ is surjective.

The diagonal of \mathcal{H} is of finite presentation (and étale and separated) so the locus $R \subseteq |\mathcal{H}|$ where the inertia stack $I_{\mathcal{H}/Y} \rightarrow \mathcal{H}$ is an isomorphism is a constructible subset (and a closed subset but we do not use this). Since Y' is the interior of R , it follows that the construction of Y' commutes with flat base change on Y and that a point $h: \text{Spec}(k) \rightarrow \mathcal{H}$ is in Y' if and only if the fibers of $I_{\mathcal{H}/Y} \rightarrow \mathcal{H}$ over h and its *generizations* have rank 1 [EGA_I, Thm. 7.3.1 and Prop. 7.3.3].

To show that a point $x: \text{Spec}(k) \rightarrow X$ is in the image of $X' \rightarrow X$, we can thus assume that Y is the spectrum of a strictly henselian local ring and that x lies in the fiber of the closed point of Y . Then by Lemma (C.2) we have that x lies in an open subscheme $Z \subseteq X$ which is finite over Y . The family $Z \rightarrow Y$ induces a morphism $Y \rightarrow Y' \subset \mathcal{H}$ which shows that x is in the image of $X' \rightarrow X$.

Finally to show (iii) it is enough to replace Y' with a quasi-compact open substack and X' with its inverse image. \square

7. STACKS WITH QUASI-FINITE DIAGONALS

In this section we show that every stack with quasi-finite diagonal has a locally quasi-finite flat presentation. The main purpose of this result is to show that the quasi-finite flat devissage, Theorem (6.1), can indeed be applied to a presentation of a stack with quasi-finite and locally separated diagonal as mentioned in Remark (6.2). We also combine this result with Theorem (6.3) and deduce that stacks with quasi-finite diagonals admit finite flat presentations étale-locally.

Theorem (7.1) (Quasi-finite presentations). *Let X be an algebraic stack with quasi-finite diagonal. Then there is a locally quasi-finite flat presentation $U \rightarrow X$ with U a scheme.*

Proof. It is enough to construct for every point $\xi \in |X|$ a locally quasi-finite flat morphism $p: U \rightarrow X$ locally of finite presentation with U a scheme such that $\xi \in p(U)$. Choose an immersion $Z \hookrightarrow X$ as in Theorem (B.2) so that $\xi \in |Z|$ and Z is an fppf gerbe over a scheme \underline{Z} . The diagonal of $Z \rightarrow \underline{Z}$ is quasi-finite, flat and locally of finite presentation. This follows from the diagram

$$\begin{array}{ccc} I_Z & \longrightarrow & Z \\ \downarrow & & \downarrow \Delta_{Z/\underline{Z}} \\ Z & \xrightarrow{\Delta_{Z/\underline{Z}}} & Z \times_{\underline{Z}} Z \end{array}$$

since the diagonal is covering in the fppf topology.

Let $V \rightarrow X$ be a flat (or smooth) presentation of X with V a scheme. Then $V \times_X Z \rightarrow Z \rightarrow \underline{Z}$ is flat. Let $\underline{\xi} \in \underline{Z}$ be the image of ξ . We will now do a standard slicing argument, cf. [EGA_{IV}, Prop. 17.16.1]. Let v be a closed point in the fiber $V_{\underline{\xi}} := V \times_X \mathcal{G}_{\underline{\xi}} = V \times_X Z \times_{\underline{Z}} \text{Spec}(k(\underline{\xi}))$ at which the fiber is Cohen–Macaulay. Let f_1, f_2, \dots, f_n be a regular sequence in $\mathcal{O}_{V_{\underline{\xi}}, v}$ such that the quotient is artinian. Since $\mathcal{O}_{V, v} \rightarrow \mathcal{O}_{V_{\underline{\xi}}, v}$ is surjective, we can lift this sequence to a sequence g_1, g_2, \dots, g_n of global sections of \mathcal{O}_V after replacing V with an open neighborhood of v .

Let $W \hookrightarrow V$ be the closed subscheme defined by the ideal (g_1, g_2, \dots, g_n) . Since f_1, f_2, \dots, f_n is regular, it follows by [EGA_{IV}, Thm. 11.3.8] that $W \times_X Z \rightarrow Z \rightarrow \underline{Z}$ is flat in a neighborhood of v . After shrinking V we can thus assume that $W \times_X Z \rightarrow Z \rightarrow \underline{Z}$ is flat. Since $W \times_X Z \rightarrow \underline{Z}$ is quasi-finite at v , we can also assume that $W \times_X Z \rightarrow Z \rightarrow \underline{Z}$ is quasi-finite after further shrinking V [EGA_{IV}, Cor. 13.1.4].

Now, since the diagonal of $Z \rightarrow \underline{Z}$ is flat and quasi-finite, it follows that $W \times_X Z \rightarrow Z$ is flat and quasi-finite. Finally, we apply [EGA_{IV}, 11.3.8 and 13.1.4] on $W \hookrightarrow V \rightarrow X$ to deduce that $W \rightarrow X$ is flat and quasi-finite in an open neighborhood of $v \in W$. \square

Theorem (7.2) (Finite flat presentations). *Let X be an algebraic stack. The following are equivalent*

- (i) *X is quasi-compact and quasi-separated with quasi-finite (resp. quasi-finite and locally separated, resp. quasi-finite and separated) diagonal.*

- (ii) *There exists a quasi-finite flat presentation $p: U \rightarrow X$ with U affine and such that p is finitely presented (resp. finitely presented and locally separated, resp. finitely presented and separated).*
- (iii) *There exists an étale (resp. representable étale, resp. representable, separated and étale) surjective morphism $X' \rightarrow X$ of finite presentation such that X' admits a finite flat presentation $V \rightarrow X'$ with V a quasi-affine scheme.*

Proof. Clearly (iii) \implies (ii). That (i) \implies (ii) follows from Theorem (7.1) and that (ii) \implies (i) follows from Lemmas (A.4) and (A.6). Assume that (i) holds and choose a quasi-finite flat presentation $p: U \rightarrow X$ with U affine as in (ii). By Theorem (6.3), there is a commutative diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow p' & \circ & \downarrow p \\ X' & \longrightarrow & X \end{array}$$

such that $X' \rightarrow X$ and $U' \rightarrow U$ are étale (resp. representable and étale, resp. representable, separated and étale) and surjective of finite presentation and $p': U' \rightarrow X'$ is finite and faithfully flat.

If X is arbitrary (resp. has locally separated diagonal), then U' is a Deligne–Mumford stack (resp. an algebraic space), so that X' has locally separated (resp. separated) diagonal by Lemma (A.4). We can thus replace X with X' and assume that X has locally separated (resp. separated) diagonal.

If X has separated diagonal, then $X' \rightarrow X$ and $U' \rightarrow U$ are separated and hence quasi-affine by Zariski’s Main Theorem [Knu71, Thm. II.6.15] and the theorem follows. \square

Remark (7.3). The proof of [SGA₁, Exp. VIII, Cor. 7.6] shows that in (iii) we can choose $V \rightarrow X' \rightarrow X$ such that V is affine. Also see [SGA₃, Exp. V, p. 270].

APPENDIX A. SEPARATION AXIOMS FOR ALGEBRAIC STACKS

A sheaf of sets F on the category of schemes **Sch** with the étale topology is an *algebraic space* if there exists a scheme X and a morphism $X \rightarrow F$ which is represented by surjective étale morphisms of schemes [RG71, Déf. 5.7.1], i.e., for any scheme T and morphism $T \rightarrow F$, the fiber product $X \times_F T$ is a scheme and $X \times_F T \rightarrow T$ is surjective and étale.

A *stack* is a category fibered in groupoids over **Sch** with the étale topology satisfying the usual sheaf condition [LMB00], or equivalently, a 2-sheaf **Sch** \rightarrow **Grpd** in the sense of Appendix D. A morphism $f: X \rightarrow Y$ of stacks is *representable* if for any scheme T and morphism $T \rightarrow Y$, the 2-fiber product $X \times_Y T$ is an algebraic space. A stack X is *algebraic* if there exists a smooth presentation, i.e., a smooth, surjective and representable morphism $U \rightarrow X$ where U is a scheme (or algebraic space).

Definition (A.1). A morphism of algebraic stacks $f: X \rightarrow Y$ is *quasi-separated* if Δ_f and Δ_{Δ_f} are quasi-compact. An algebraic stack X is *quasi-separated* if $X \rightarrow \mathrm{Spec}(\mathbb{Z})$ is quasi-separated. A morphism of algebraic

stacks is of *finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

Recall that in [LMB00] algebraic stacks are by definition quasi-separated and have separated diagonals. In the remainder of this appendix we give criteria for when this is the case. An important example of a stack with non-separated diagonal is the stack of log structures [Ols03]. On the other hand, this stack has at least *locally separated diagonal*.

Definition (A.2). Let $f: X \rightarrow Y$ be a representable morphism. We say that f is *locally separated* if Δ_f is an immersion.

In particular, every Deligne–Mumford stack has locally separated diagonal.

Lemma (A.3). Let X be an algebraic stack. The following are equivalent:

- (i) Δ_X is separated (resp. locally separated, resp. quasi-separated).
- (ii) The inertia stack $I_X \rightarrow X$ is separated (resp. locally separated, resp. quasi-separated).
- (iii) The unit section $X \rightarrow I_X$ of the inertia stack is a closed immersion (resp. an immersion, resp. quasi-compact).

Lemma (A.4). Let $f: X \rightarrow Y$ be a faithfully flat morphism, locally of finite presentation, between algebraic stacks.

- (i) If f and Δ_X are separated then so is Δ_Y .
- (ii) If f is representable and f and Δ_X are locally separated, then so is Δ_Y .

Proof. This follows from the cartesian diagram

$$\begin{array}{ccccccc} X & \hookrightarrow & I_X & \longrightarrow & I_Y \times_Y X & \longrightarrow & X \\ \downarrow & & & \square & \downarrow & \square & \downarrow \\ Y & \hookrightarrow & I_Y & \longrightarrow & Y & \longrightarrow & Y \end{array}$$

Indeed, it is enough to show that the unit section $Y \hookrightarrow I_Y$ of the inertia stack is a closed immersion (resp. an immersion). By fppf-descent (or by noting that f is universally open) it is enough to show that the morphisms $X \rightarrow I_X$ and $I_X \rightarrow I_Y \times_Y X$ are proper (resp. immersions). This is the case since the first map is the double diagonal of X and the second map is a pull-back of the diagonal of f . \square

Lemma (A.5). Let $f: X \rightarrow Y$ be a morphism of stacks.

- (i) If X and Δ_Y are quasi-compact then so is f .
- (ii) If Δ_X , and Δ_{Δ_Y} are quasi-compact then so is Δ_f .
- (iii) If Δ_{Δ_X} is quasi-compact then so is Δ_{Δ_f} .

In particular, if X and Y are quasi-separated then so is f .

Lemma (A.6). Let $f: X \rightarrow Y$ be a surjective morphism of algebraic stacks.

- (i) If X is quasi-compact then so is Y .
- (ii) If f and Δ_X are quasi-compact, then so is Δ_Y .
- (iii) If f , Δ_f and Δ_{Δ_X} are quasi-compact, then so is Δ_{Δ_Y} .

In particular, if f is quasi-compact and quasi-separated and X is quasi-separated then Y is quasi-separated.

Proof. (i) Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & \circ & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

with U and V schemes and such that the vertical morphisms are smooth and surjective. If X is quasi-compact then we can choose U quasi-compact and hence also V .

(ii)–(iii) The latter statements follows from (i) and the commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \times X \\ f \downarrow & \circ & \downarrow f \times f \\ Y & \xrightarrow{\Delta_Y} & Y \times Y \end{array} \quad \text{and} \quad \begin{array}{ccccccc} X & \xrightarrow{\Delta_X} & I_X & \xrightarrow{g} & I_Y \times_Y X & \longrightarrow & X \\ \downarrow f & & \square & & \downarrow & \square & \downarrow f \\ Y & \xrightarrow{\Delta_Y} & I_Y & \longrightarrow & Y & & Y \end{array}$$

since g is a pull-back of Δ_f . \square

APPENDIX B. ALGEBRAICITY OF POINTS ON QUASI-SEPARATED STACKS

Let X be a quasi-separated algebraic stack. In this section we show that every point on X is algebraic, i.e., that for every point $\xi \in |X|$ there is an algebraic stack \mathcal{G}_ξ and a monomorphism $\mathcal{G}_\xi \hookrightarrow X$ with image ξ such that \mathcal{G}_ξ is an fppf-gerbe over the spectrum of a field $k(\xi)$. In fact, we proof the stronger statement that \mathcal{G}_ξ is the generic fiber of a gerbe $Z \rightarrow \underline{Z}$ where $Z \hookrightarrow X$ is an immersion and \underline{Z} is an integral scheme. It is also enough to assume that X has quasi-compact (but not necessarily quasi-separated) diagonal.

When X is a locally noetherian stack, this result is shown in [LMB00, §11] although the definition of algebraic point is slightly wrong. The error in [LMB00, Def. 11.2] is the assertion that if $x: \text{Spec}(k) \rightarrow X$ is *any* representative of ξ and $\text{Spec}(k) \twoheadrightarrow \mathcal{G}_x \hookrightarrow X$ is its epi-mono factorization as fppf stacks, then \mathcal{G}_x is independent of the choice of representative x . This assertion is not correct unless restricted to fields k which are finite over the residue field $k(\xi)$, the reason being that non-finite field extensions are not covering in the fppf topology. It is possible that the assertion is valid with respect to the *fppc topology* but that approach opens up other difficulties as the epi-mono factorization in the fpqc topology a priori depends on the choice of universe.

To obtain the algebraicity in full generality we begin with a generic flatness result due to Raynaud and Gruson.

Theorem (B.1) (Generic flatness). *Let Y be an integral scheme. Let X be an algebraic stack and let $f: X \rightarrow Y$ be a morphism of finite type. Then there exists an open dense subscheme $Y_0 \subseteq Y$ such that $f|_{Y_0}$ is flat and locally of finite presentation.*

Proof. We can replace Y with an open dense affine subscheme. We can also replace X with a presentation and assume that X is affine.

Choose a finitely presented affine Y -scheme \overline{X} and a closed immersion $j: X \hookrightarrow \overline{X}$ over Y . Let $U \subseteq X$ be the locus where f is flat and let $\overline{U} \subseteq \overline{X}$ be the locus where $j_*\mathcal{O}_X$ is flat over Y . Then $\overline{U} = \overline{X} \setminus j(X \setminus U)$ so that $U = j^{-1}(\overline{U})$. According to Raynaud and Gruson [RG71, Thm. 3.4.6], the subset $\overline{U} \subseteq \overline{X}$ is open and $(j_*\mathcal{O}_X)|_{\overline{U}}$ is an $\mathcal{O}_{\overline{U}}$ -module of finite presentation. Equivalently, we have that $U \subseteq X$ is open and $f|_U$ is of finite presentation.

It remains to find an open dense subset $V \subseteq Y$ such that $f^{-1}(V)$ is contained in U . We let $V = Y \setminus \overline{f(X \setminus U)}$ which suffices if we can show that V is not the empty set. Since f is quasi-compact, it follows that $f(X \setminus U)$ is pro-constructible and hence that $\overline{f(X \setminus U)}$ coincides with the specialization of $f(X \setminus U)$ [EGA_I, Cor. 7.3.2]. Since f is trivially flat at the generic point of Y , it follows that V is non-empty. \square

Theorem (B.2) (Algebraicity of points). *Let X be an algebraic stack with quasi-compact diagonal. Let $\xi \in |X|$ be a point. Then there is a quasi-compact immersion $Z \hookrightarrow X$ such that*

- (i) $\xi \in |Z|$,
- (ii) *The inertia stack $I_Z \rightarrow Z$ is flat and locally of finite presentation,*
- (iii) *The stack Z is an fppf-gerbe over an affine scheme \underline{Z} . The structure morphism $\pi: Z \rightarrow \underline{Z}$ is faithfully flat and locally of finite presentation. The scheme \underline{Z} is integral with generic point $\underline{\xi} = \pi(\xi)$.*

In particular, $\xi \in |X|$ is algebraic with residual gerbe $\mathcal{G}_\xi = \pi^{-1}(\underline{\xi})$ and residual field $k(\underline{\xi})$ and the monomorphism $\mathcal{G}_\xi \hookrightarrow X$ is quasi-affine.

Proof. We can replace X with the reduced closed substack $\overline{\{\xi\}}$ so that $|X|$ is irreducible. To show (ii), it is then enough to show that $I_X \rightarrow X$ is flat and locally of finite presentation over a non-empty quasi-compact open subset $Z \subseteq X$. Let $p: U \rightarrow X$ be a smooth presentation with U a scheme. We can replace U with an affine non-empty open subscheme and X with its image and assume that U is affine. Let $x: \text{Spec}(k) \rightarrow X$ be a representative of ξ . Since X has quasi-compact diagonal, it follows that x is quasi-compact. Thus $x^{-1}(U) \rightarrow U$ is quasi-compact so that $x^{-1}(U)$ is an algebraic space of finite type over $\text{Spec}(k)$. Let $W \rightarrow x^{-1}(U)$ be an étale presentation with W an affine scheme of finite type over $\text{Spec}(k)$.

As p is open we have that U is the closure of the image of W . As W has a finite number of irreducible components, so has U . We can thus replace U by an open non-empty irreducible quasi-compact subscheme and assume that U is an integral scheme. It now follows from Theorem (B.1) that $I_X \times_X U \rightarrow U$ is flat and locally of finite presentation over an open dense subscheme $U_0 \subseteq U$. We let $Z = p(U_0)$ and (ii) follows by flat descent.

Now, as $I_Z \rightarrow Z$ is flat and locally of finite presentation, we have that the fppf-sheafification \underline{Z} of Z is an algebraic space and that $Z \rightarrow \underline{Z}$ is faithfully flat and locally of finite presentation [LMB00, Cor. 10.8]. Moreover as the diagonal of Z is quasi-compact and $\Delta_{Z/\underline{Z}}$ is surjective, it follows that the diagonal of \underline{Z} is quasi-compact, i.e., that \underline{Z} is a quasi-separated algebraic

space. After replacing \underline{Z} with a dense open we can thus assume that \underline{Z} is a *scheme*.

Since Z is reduced with generic point ξ , we have that \underline{Z} is reduced with generic point $\underline{\xi}$. We may thus replace \underline{Z} with an open dense subscheme so that \underline{Z} becomes affine. \square

APPENDIX C. TWO LEMMAS ON ALGEBRAIC SPACES

In this section we state two lemmas on algebraic spaces that likely are well-known to experts.

Lemma (C.1). *Let X be an algebraic space and let $p: X' \rightarrow X$ be a finite flat presentation by an affine (resp. quasi-affine) scheme X' . Then X is an affine (resp. quasi-affine) scheme.*

Proof. Every fiber of p has an affine open neighborhood by [EGA_{II}, Cor. 4.5.4]. Hence X is an affine scheme (resp. a scheme) [SGA₃, Exp. V, Thm. 4.1]. If X' is quasi-affine then so is X by [EGA_{II}, Cor. 6.6.3]. \square

Lemma (C.2). *Let $S = \operatorname{Spec}(A)$ be strictly local, i.e., let A be a strictly henselian local ring. Let X be an algebraic space and let $X \rightarrow S$ be locally quasi-finite and locally separated. Let $\bar{x}: \operatorname{Spec}(k) \rightarrow X$ be a geometric point over the closed point $s \in S$ and let $\mathcal{O}_{X,\bar{x}}$ denote the strictly local ring. Then $g: \operatorname{Spec}(\mathcal{O}_{X,\bar{x}}) \rightarrow X$ is an open immersion and $\operatorname{Spec}(\mathcal{O}_{X,\bar{x}}) \rightarrow S$ is finite.*

Proof. The lemma is well-known for schemes [EGA_{IV}, Thm. 18.5.11] (it is then enough to assume that A is henselian). Let U be a scheme and let $U \rightarrow X$ be an étale presentation. Then \bar{x} lifts to U so that $Z = \operatorname{Spec}(\mathcal{O}_{X,\bar{x}})$ is an open subscheme of U and $Z \rightarrow S$ is finite. It follows that $g: Z \rightarrow X$ is étale. The scheme $Z \times_S Z$ is local. By assumption, the morphism $Z \times_X Z \rightarrow Z \times_S Z$ is an immersion and as the closed point of $Z \times_S Z$ lies in $Z \times_X Z$ it follows that $Z \times_X Z \rightarrow Z \times_S Z$ is a closed immersion. In particular, $Z \times_X Z$ is finite over Z so that $Z \rightarrow g(Z)$ is finite and étale. Since g has rank 1 at x it follows that $Z \rightarrow g(Z)$ is an isomorphism so that g is an open immersion. \square

APPENDIX D. 2-SHEAVES ON THE CATEGORY OF ALGEBRAIC STACKS

In this appendix we define 2-sheaves on the 2-category of algebraic stacks with the étale topology. A 2-presheaf (resp. a 2-sheaf) is a generalization of the notion of a fibered category (resp. a stack) which allows the base category (resp. site) to be a 2-category. We have chosen to describe 2-presheaves in terms of 2-functors and not in terms of fibered 2-categories as this appears to be the simplest description. Similarly, one usually describe 1-sheaves as ordinary functors and not as fibered categories with fibers equivalent to discrete categories. There are essentially three different ways to describe the sheaf condition: with sieves, (semi-)simplicial objects or classical descent data. Our presentation takes the classical approach.

The general theory of 2-sheaves has been developed by R. Street in two papers. The first paper [Str82b] treats the, from our perspective, less interesting case where *all* notions are strict. The second paper [Str82a] briefly treats the non-strict case (which generalizes fibered categories and stacks)

but the proofs have to be copied and modified from the first paper. We have therefore decided to make the following presentation independent of these two papers. To further simplify the discussion, the results are stated for *strict* 2-presheaves although the results remain valid for arbitrary 2-presheaves. This latter notion of strictness should not be confused with the all-encompassing strictness imposed in [Str82b]. Moreover, every 2-presheaf is equivalent (but not isomorphic) to a strict 2-presheaf by the bicategorical Yoneda lemma so we do not lose anything by limiting ourselves to the 2-category of strict 2-presheaves.

Definition (D.1). A 2-category is a category \mathbf{C} enriched in categories, i.e., for every pair of objects (X, Y) in \mathbf{C} we have a category $\mathbf{Hom}_{\mathbf{C}}(X, Y)$. The objects (resp. arrows) of $\mathbf{Hom}_{\mathbf{C}}(X, Y)$ are called 1-morphisms (resp. 2-morphisms). We say that \mathbf{C} is a $(2, 1)$ -category if every 2-morphism is invertible, i.e., if $\mathbf{Hom}_{\mathbf{C}}(X, Y)$ is a groupoid for every X, Y .

The standard example of a 2-category is the 2-category \mathbf{Cat} of categories, functors and natural transformations. Similarly, the standard example of a $(2, 1)$ -category is the full subcategory $\mathbf{Grpd} \subseteq \mathbf{Cat}$ of groupoids. The other important example of a $(2, 1)$ -category is the $(2, 1)$ -category of algebraic stacks \mathbf{Stack} . All these categories have 2-fiber products.

Definition (D.2). A *strict 2-functor* $F: \mathbf{C} \rightarrow \mathbf{D}$ between 2-categories consists of

- (i) a map $F: \text{ob } \mathbf{C} \rightarrow \text{ob } \mathbf{D}$,
- (ii) for every pair of objects $X, Y \in \text{ob } \mathbf{C}$, a functor

$$F(X, Y): \mathbf{Hom}_{\mathbf{C}}(X, Y) \rightarrow \mathbf{Hom}_{\mathbf{D}}(F(X), F(Y)),$$

such that

- (a) $F(\text{id}_X) = \text{id}_{F(X)}$ for every $x \in \text{ob } \mathbf{C}$,
- (b) for every $X, Y, Z \in \mathbf{C}$, the diagram

$$\begin{array}{ccc} \mathbf{Hom}_{\mathbf{C}}(Y, Z) \times \mathbf{Hom}_{\mathbf{C}}(X, Y) & \xrightarrow{\circ} & \mathbf{Hom}_{\mathbf{C}}(X, Z) \\ F(Y, Z) \times F(X, Y) \downarrow & & \downarrow F(X, Z) \\ \mathbf{Hom}_{\mathbf{D}}(FY, FZ) \times \mathbf{Hom}_{\mathbf{D}}(FX, FY) & \xrightarrow{\circ} & \mathbf{Hom}_{\mathbf{D}}(FX, FZ) \end{array}$$

is strictly commutative.

A 2-functor is defined similarly but instead of requiring that the functor respects identities and composition as in (a) and (b), the data for a 2-functor include 2-isomorphisms $\text{id}_{F(X)} \Rightarrow F(\text{id}_X)$ and natural isomorphisms between the two functors $\mathbf{Hom}(Y, Z) \times \mathbf{Hom}(X, Y) \rightarrow \mathbf{Hom}(FX, FZ)$. These isomorphisms are then required to satisfy natural coherence conditions. In some literature 2-functors are called *pseudofunctors* and strict 2-functors are simply called 2-functors.

Definition (D.3). Let \mathbf{C} be a 2-category. A (strict) 2-presheaf on \mathbf{C} is a (strict) 2-functor $\mathbf{F}: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Cat}$.

The general definition of topologies and 2-sheaves on 2-categories can be found in [Str82a]. In the remainder of this appendix we will give a concrete description of strict 2-sheaves on the $(2, 1)$ -category **Stack** of algebraic stacks with the étale topology. Fix a strict 2-presheaf $\mathbf{F}: \mathbf{Stack}^{\text{op}} \rightarrow \mathbf{Cat}$.

Definition (D.4). A family of morphisms $(p_\alpha: X_\alpha \rightarrow X)$ in **Stack** is *covering in the étale topology* if the p_α 's are smooth and $\coprod_\alpha X_\alpha \rightarrow X$ is surjective.

Note that the p_α 's need not be representable.

(D.5) Let $(p_\alpha: X_\alpha \rightarrow X)_\alpha$ be a family of morphisms in **Stack**, let $X_{\alpha\beta} = X_\alpha \times_X X_\beta$ be a 2-fiber product with projections $\pi_1: X_{\alpha\beta} \rightarrow X_\alpha$ and $\pi_2: X_{\alpha\beta} \rightarrow X_\beta$ and 2-isomorphism $p_\alpha \circ \pi_1 \Rightarrow p_\beta \circ \pi_2$. This induces a natural isomorphism of functors

$$\pi_1^* p_\alpha^* = (p_\alpha \circ \pi_1)^* \cong (p_\beta \circ \pi_2)^* = \pi_2^* p_\beta^*: \mathbf{F}(X) \rightarrow \mathbf{F}(X_{\alpha\beta}).$$

In particular, if $f: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism in $\mathbf{F}(X)$, then we obtain a commutative diagram

$$\begin{array}{ccc} \pi_1^* p_\alpha^* \mathcal{F} & \xrightarrow{\pi_1^* p_\alpha^* f} & \pi_1^* p_\alpha^* \mathcal{G} \\ \downarrow \cong & & \downarrow \cong \\ \pi_2^* p_\beta^* \mathcal{F} & \xrightarrow{\pi_2^* p_\beta^* f} & \pi_2^* p_\beta^* \mathcal{G}. \end{array}$$

Definition (D.6). A family of morphisms $(p_\alpha: X_\alpha \rightarrow X)_\alpha$ in **Stack** is a *family of descent* for \mathbf{F} if for every $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$ the sequence

$$\text{Hom}_{\mathbf{F}(X)}(\mathcal{F}, \mathcal{G}) \xrightarrow{(p_\alpha^*)} \prod_\alpha \text{Hom}_{\mathbf{F}(X_\alpha)}(\mathcal{F}_\alpha, \mathcal{G}_\alpha) \xrightleftharpoons[\pi_2^*]{\pi_1^*} \prod_{\alpha\beta} \text{Hom}_{\mathbf{F}(X_{\alpha\beta})}(\mathcal{F}_{\alpha\beta}, \mathcal{G}_{\alpha\beta})$$

is exact where $\mathcal{F}_\alpha = p_\alpha^* \mathcal{F}$, $\mathcal{F}_{\alpha\beta} = \pi_1^* p_\alpha^* \mathcal{F}$ and we implicitly have used the canonical isomorphism $\pi_1^* p_\alpha^* \mathcal{F} \rightarrow \pi_2^* p_\beta^* \mathcal{F}$ (and similarly for \mathcal{G}).

(D.7) Cocycle condition — Let $(p_\alpha: X_\alpha \rightarrow X)_\alpha$ and π_1, π_2 be as in (D.5). Given an object $\mathcal{F} \in \mathbf{F}(X)$ we saw that we obtained a canonical isomorphism $\psi_{\alpha\beta}: \pi_1^* \mathcal{F}_\alpha \xrightarrow{\cong} \pi_2^* \mathcal{F}_\beta$ in $\mathbf{F}(X_{\alpha\beta})$ where $\mathcal{F}_\alpha = p_\alpha^* \mathcal{F}$. The isomorphism $\psi_{\alpha\beta}$ satisfies the cocycle condition, i.e., the following diagram in $\mathbf{F}(X_{\alpha\beta\gamma})$

$$\begin{array}{ccccc} & \pi_{12}^* \pi_2^* \mathcal{F}_\beta & \xrightarrow[\cong]{\text{can}} & \pi_{23}^* \pi_1^* \mathcal{F}_\beta & \\ \pi_{12}^* (\psi_{\alpha\beta}) \nearrow & & & & \searrow \pi_{23}^* (\psi_{\beta\gamma}) \\ \pi_{12}^* \pi_1^* \mathcal{F}_\alpha & & \circ & & \pi_{23}^* \pi_2^* \mathcal{F}_\gamma \\ & \nwarrow \text{can} & & \nwarrow \text{can} & \\ & \pi_{31}^* \pi_2^* \mathcal{F}_\alpha & \xleftarrow[\cong]{\pi_{31}^* (\psi_{\gamma\alpha})} & \pi_{31}^* \pi_1^* \mathcal{F}_\gamma & \end{array}$$

commutes. Here $X_{\alpha\beta\gamma} = X_\alpha \times_X X_\beta \times_X X_\gamma$ is a 2-fiber product, $\pi_{ij}: X_{\alpha_1\alpha_2\alpha_3} \rightarrow X_{\alpha_i\alpha_j}$ is the projection onto the i^{th} and j^{th} factors and the maps denoted with “can” are canonical isomorphisms.

Definition (D.8). Let $(\mathcal{F}_\alpha)_\alpha \in \prod_\alpha \mathbf{F}(X_\alpha)$. A *descent datum* for $(\mathcal{F}_\alpha)_\alpha$ is a collection of isomorphisms $\psi_{\alpha\beta}: \pi_1^* \mathcal{F}_\alpha \rightarrow \pi_2^* \mathcal{F}_\beta$ in $\mathbf{F}(X_{\alpha\beta})$ satisfying the cocycle condition.

Definition (D.9). We let $\mathbf{F}((p_\alpha)_\alpha) = \mathbf{F}((X_\alpha \rightarrow X)_\alpha)$ be the category with

- objects: pairs $((\mathcal{F}_\alpha), (\psi_{\alpha\beta}))$ of an object $(\mathcal{F}_\alpha) \in \prod_\alpha \mathbf{F}(X_\alpha)$ equipped with a descent datum $(\psi_{\alpha\beta})$.
- morphisms $((\mathcal{F}_\alpha), (\psi_{\alpha\beta})) \rightarrow ((\mathcal{G}_\alpha), (\theta_{\alpha\beta}))$: a morphism $(f_\alpha): (\mathcal{F}_\alpha) \rightarrow (\mathcal{G}_\alpha)$ in $\prod_\alpha \mathbf{F}(X_\alpha)$ such that

$$\begin{array}{ccc} \pi_1^* \mathcal{F}_\alpha & \xrightarrow{\pi_1^* f_\alpha} & \pi_1^* \mathcal{G}_\alpha \\ \downarrow \psi_{\alpha\beta} & & \downarrow \theta_{\alpha\beta} \\ \pi_2^* \mathcal{F}_\beta & \xrightarrow{\pi_2^* f_\beta} & \pi_2^* \mathcal{G}_\beta \end{array}$$

commutes for all pairs (α, β) .

There is a natural functor $(p_\alpha)_D^*: \mathbf{F}(X) \rightarrow \mathbf{F}((p_\alpha)_\alpha)$ taking an object $\mathcal{F} \in \mathbf{F}(X)$ onto $(p_\alpha^* \mathcal{F})$ equipped with the induced descent datum and taking a morphism $\mathcal{F} \rightarrow \mathcal{G}$ onto $(p_\alpha^* \mathcal{F} \rightarrow p_\alpha^* \mathcal{G})$. The functor $(p_\alpha)_D^*$ is fully faithful if and only if (p_α) is a family of descent for \mathbf{F} .

Definition (D.10). A family of descent (p_α) is a family of *effective descent* for \mathbf{F} if $(p_\alpha)_D^*: \mathbf{F}(X) \rightarrow \mathbf{F}((p_\alpha)_\alpha)$ is an equivalence of categories.

Definition (D.11). A 2-presheaf $\mathbf{F}: \mathbf{Stack}^{\text{op}} \rightarrow \mathbf{Cat}$ is a 2-sheaf in the étale topology if every covering family $(X_\alpha \rightarrow X)_\alpha$ in the étale topology is of effective descent for \mathbf{F} .

Definition (D.12). We let \mathbf{Stack}' denote the subcategory of \mathbf{Stack} with all objects but with only smooth morphisms. For an algebraic stack X , we let \mathbf{Stack}_X denote the 2-category of morphisms $Z \rightarrow X$. We let $\mathbf{Stack}_{\text{ét}/X} \subset \mathbf{Stack}_{\text{sm}/X} \subset \mathbf{Stack}_X$ denote the full 2-subcategories of étale and smooth morphisms and we let $\mathbf{Stack}_{\text{repr}/X} \subset \mathbf{Stack}_X$ denote the full 1-category of representable morphisms. We say that a family of morphisms in \mathbf{Stack}_X is covering if its image in \mathbf{Stack} is covering. We say that a 2-presheaf on any of these categories is a 2-sheaf if every covering family is of effective descent.

If \mathbf{F} is a 2-sheaf on \mathbf{Stack} (resp. \mathbf{Stack}') then the restricted 2-presheaf on \mathbf{Stack}_X (resp. $\mathbf{Stack}_{\text{sm}/X}$) is a 2-sheaf for any stack X . In particular, the restriction to $\mathbf{Stack}_{\text{ét}/X}$ is a 2-sheaf.

By the comparison lemma for 2-sheaves [Str82b, Thm. 3.8] restriction along $\mathbf{Stack}_{\text{repr}/X} \subset \mathbf{Stack}_X$ induces a 2-equivalence between the 2-category of 2-sheaves on \mathbf{Stack}_X and the 2-category of 2-sheaves on $\mathbf{Stack}_{\text{repr}/X}$, or equivalently, the 2-category of stacks on $\mathbf{Stack}_{\text{repr}/X}$. The following result is essentially a reformulation of the comparison lemma (also see [Gir64, Prop. 10.10]).

Proposition (D.13). Let $\mathbf{F}: \mathbf{Stack}^{\text{op}} \rightarrow \mathbf{Cat}$ be a strict 2-presheaf. Then \mathbf{F} is a 2-sheaf if and only if the following two conditions hold.

- (i) For every algebraic stack X and every surjective smooth morphism $p: U \rightarrow X$ such that U is an algebraic space, we have that p is of effective descent.
- (ii) For every family of algebraic spaces (X_α) the natural functor

$$\mathbf{F}\left(\coprod_{\alpha} X_{\alpha}\right) \rightarrow \prod_{\alpha} \mathbf{F}(X_{\alpha})$$

is an equivalence of categories.

Proof. The functor in (ii) is an equivalence if and only if for every algebraic space $X = \coprod X_{\alpha}$, the family $(j_{\alpha}: X_{\alpha} \hookrightarrow X)$ is of effective descent, cf. [Gir64, Prop. 9.24]. Thus, the two conditions are necessary. Moreover, if (ii) holds, then for every covering family $(p_{\alpha}: X_{\alpha} \rightarrow X)$ such that the X_{α} 's are algebraic spaces, the natural functor

$$\mathbf{F}\left(\coprod_{\alpha} p_{\alpha}: \coprod_{\alpha} X_{\alpha} \rightarrow X\right) \rightarrow \mathbf{F}((p_{\alpha}: X_{\alpha} \rightarrow X))$$

is an equivalence.

To show that the conditions are sufficient, assume that (i) and (ii) holds and let $(p_{\alpha}: X_{\alpha} \rightarrow X)$ be a covering family. For every α choose a smooth presentation $q_{\alpha}: U_{\alpha} \rightarrow X_{\alpha}$ so that we obtain 2-commutative diagrams

$$\begin{array}{ccccc} U_{\alpha\beta} := U_{\alpha} \times_X U_{\beta} & \xrightarrow{\pi_1} & U_{\alpha} & & \\ \downarrow q_{\alpha\beta} & & \downarrow q_{\alpha} & & \\ X_{\alpha\beta} := X_{\alpha} \times_X X_{\beta} & \xrightarrow{\pi_1} & X_{\alpha} & \xrightarrow{p_{\alpha}} & X \end{array}$$

and similarly for π_2 . With the usual choice of the 2-fiber product $X_{\alpha\beta}$, we can even assume that the diagram is strictly commutative. There is a natural functor

$$Q: \mathbf{F}((p_{\alpha})_{\alpha}) \rightarrow \mathbf{F}((p_{\alpha} \circ q_{\alpha})_{\alpha})$$

taking an object $((\mathcal{F}_{\alpha}), (\psi_{\alpha\beta}))$ to $((q_{\alpha}^* \mathcal{F}_{\alpha}), (q_{\alpha\beta}^* \psi_{\alpha\beta}))$ so that $(p_{\alpha} \circ q_{\alpha})_D^* = Q \circ (p_{\alpha})_D^*$.

Since q_{α} and $q_{\alpha\beta}$ are of descent it follows that Q is fully faithful. Indeed, that Q is faithful is immediate from the faithfulness of q_{α}^* . To see that Q is full, let $((\mathcal{F}_{\alpha}), (\psi_{\alpha\beta}))$ and $((\mathcal{G}_{\alpha}), (\theta_{\alpha\beta}))$ be objects of $\mathbf{F}(p_{\alpha})$ and let $(g_{\alpha}): (q_{\alpha}^* \mathcal{F}_{\alpha}) \rightarrow (q_{\alpha}^* \mathcal{G}_{\alpha})$ be a morphism in $\mathbf{F}(p_{\alpha} \circ q_{\alpha})$. Since $(q_{\alpha})_D^*$ is full g_{α} descends to a map $(f_{\alpha}): (\mathcal{F}_{\alpha}) \rightarrow (\mathcal{G}_{\alpha})$. That this map is compatible with the descent data ψ and θ follows from the faithfulness of $q_{\alpha\beta}^*$.

As $(p_{\alpha} \circ q_{\alpha})$ is of effective descent, it follows that Q is essentially surjective and hence an equivalence of categories. It follows that (p_{α}) is a family of effective descent and that \mathbf{F} is a 2-sheaf. \square

More generally, the proposition holds for non-strict 2-presheaves. In this case, we only have a natural isomorphism $(p_{\alpha} \circ q_{\alpha})_D^* \cong Q \circ (p_{\alpha})_D^*$ in the proof.

APPENDIX E. EXAMPLES OF 2-SHEAVES

In this appendix we show that the 2-presheaf $\mathbf{Hom}(-, Y)$ of morphisms to a given stack Y and the 2-presheaf $\mathbf{QCoh}(-)$ of quasi-coherent sheaves are 2-sheaves. For simplicity, we only treat the restriction of \mathbf{QCoh} to the full subcategory $\mathbf{Stack}' \subset \mathbf{Stack}$ so that \mathbf{QCoh} is a *strict* 2-sheaf.

Let Y be an algebraic stack. There is a strict 2-functor

$$\mathbf{Hom}(-, Y): \mathbf{Stack} \rightarrow \mathbf{Grpd}$$

which takes an algebraic stack X to the groupoid $\mathbf{Hom}(X, Y)$.

Theorem (E.1). *The strict 2-presheaf $\mathbf{Hom}(-, Y): \mathbf{Stack}^{\text{op}} \rightarrow \mathbf{Grpd}$ is a 2-sheaf.*

Proof. Let us verify the conditions of Proposition (D.13). Part (i) is the description of $\mathbf{Hom}(X, Y)$ given in [LMB00, pf. of Prop. 4.18] and part (ii) is the definition of the coproduct in the 2-category of stacks. \square

There is a strict 2-functor

$$\mathbf{Mod}_{\text{cart}}(\mathcal{O}_-): \mathbf{Stack}'^{\text{op}} \rightarrow \mathbf{Cat}$$

taking an algebraic stack X to the category $\mathbf{Mod}_{\text{cart}}(\mathcal{O}_X)$ of cartesian lisse-étale \mathcal{O}_X -modules. For a smooth morphism $f: X' \rightarrow X$ the pull-back $f^*: \mathbf{Mod}_{\text{cart}}(\mathcal{O}_X) \rightarrow \mathbf{Mod}_{\text{cart}}(\mathcal{O}_{X'})$ is defined by restriction along the functor $\text{Lis-ét}(X') \rightarrow \text{Lis-ét}(X)$. This defines f^* uniquely (not merely up to isomorphism) so that $\mathbf{Mod}_{\text{cart}}(\mathcal{O}_-)$ is indeed a strict 2-presheaf. We let

$$\mathbf{QCoh}(-): \mathbf{Stack}'^{\text{op}} \rightarrow \mathbf{Cat}$$

be the sub-2-presheaf of quasi-coherent modules.

Theorem (E.2). *The strict 2-presheaf $\mathbf{QCoh}(-): \mathbf{Stack}'^{\text{op}} \rightarrow \mathbf{Cat}$ is a 2-sheaf.*

Proof. We will verify the conditions of Proposition (D.13) for \mathbf{QCoh} . Condition (i) for the 2-presheaf $\mathbf{Mod}_{\text{cart}}(\mathcal{O}_-)$ is [Ols07, Lem. 4.5] and that condition (ii) holds is obvious.

It remains to verify that if X is an algebraic stack, if \mathcal{F} is a cartesian \mathcal{O}_X -module, and if $p: U \rightarrow X$ is a smooth presentation, then \mathcal{F} is a quasi-coherent \mathcal{O}_X -module if and only if $p^*\mathcal{F}$ is a quasi-coherent \mathcal{O}_U -module. This is [LMB00, Prop. 13.2.1]. \square

It is also not difficult to show that the 2-presheaves of: big-étale sheaves, lisse-étale sheaves, cartesian lisse-étale sheaves, constructible sheaves, \mathcal{A} -modules, cartesian \mathcal{A} -modules; are all 2-sheaves. Here \mathcal{A} denotes a flat lisse-étale sheaf of rings. Another important 2-sheaf is the 2-presheaf $\mathbf{Stack}_{\text{repr}}$ which takes a stack X to the 1-category $\mathbf{Stack}_{\text{repr}/X}$ of representable morphisms $Z \rightarrow X$. This is a subsheaf of the 2-sheaf of big-étale sheaves. That $\mathbf{Stack}_{\text{repr}}$ is a 2-sheaf follows from [LMB00, Cor. 10.5].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, 970 EVANS HALL #3840, BERKELEY, CA 94720-3840 USA

E-mail address: dary@math.berkeley.edu